J. Phys. A: Math. Gen. **33** (2000) 5159–5169. Printed in the UK PII: S0305-4470(00)12079-7

Classical information entropy and phase distributions of optical fields

A Miranowicz†‡, H Matsueda† and M R B Wahiddin§

† Department of Information Science, Kochi University, 2-5-1 Akebono-cho, Kochi 780-8520, Japan

‡ Nonlinear Optics Division, Institute of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland

§ Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia

Received 23 February 2000, in final form 19 May 2000

Abstract. The Wehrl phase distribution is defined as a phase density of the Wehrl classical information entropy. The new measure is applied to describe the quantum phase properties of some optical fields including Fock states, coherent and squeezed states, and superposition of chaotic and coherent fields. The Wehrl phase distribution is compared with both the conventional Wehrl entropy and Husimi phase distribution (the marginal Husimi Q-function). It is shown that the Wehrl phase distribution is a good measure of the phase-space uncertainty (noise), phase locking and phase bifurcation effects. It is also demonstrated that the Wehrl phase distribution properly describes phase randomization processes, and thus can be used in a description of the quantum optical phase.

1. Introduction

Quantum information-theoretic entropy is one of the most fundamental concepts of quantum physics [1]. Quantum entropy, as a natural generalization of the Boltzmann classical entropy, was proposed by von Neumann [2]. It has been applied, in particular, as a measure of quantum entanglement, quantum decoherence, photocount statistics, quantum optical correlations, purity of states, quantum noise or accessible information in quantum measurement (the capacity of the quantum channel).

Classical information-theoretic entropy associated with quantum fields was introduced by Wehrl [3] in terms of the Glauber coherent states and the Husimi Q-function [4]. A rigorous proof that the von Neumann entropy tends to the Wehrl entropy in the classical limit $\hbar \to 0$ was given by Beretta [5]. The von Neumann quantum entropy can be expanded in a power series of classical entropies. As was shown explicitly by Peřinová *et al* [6], the first term of this expansion is the Wehrl entropy. Bužek *et al* [7] related the von Neumann entropy to the sampling entropies based on an operational approach to a phase-space measurement. They also identified the Wehrl entropy as a particular example of the sampling entropy, when the quantum ruler is represented by coherent states. Other aspects of the relation between classical and quantum mechanical entropies have also been extensively studied (see, e.g., excellent surveys by Wehrl [8], and Ohya and Petz [1] with references included therein).

The Wehrl entropy has been successfully applied in a description of different properties of quantum optical fields. In particular, it has been shown explicitly that the Wehrl entropy is a useful measure of phase-space uncertainty (quantum noise, phase-space localization, wavepacket spreading) [7, 9–13], quantum interference [9], decoherence [10, 14, 15],

0305-4470/00/295159+11$30.00

ionization [13], squeezing [15–17], Schrödinger cat formation [9,12,18] or, in general, splitting of the Q-function [14]. Wehrl entropy for the common quantum states of light has been analysed in detail, e.g., by Lee [16], Peřinová *et al* [6], Orłowski *et al* [12, 14, 17, 18], and Bužek *et al* [7, 9].

The von Neumann entropy becomes zero for all pure states, and thus cannot be used in discriminating them. Paradoxically, the Wehrl classical entropy is more sensitive in distinguishing states than the von Neumann quantum entropy, since it depends on the choice of pure states. However, there are properties of quantum fields, including phase properties, which are not precisely enough described by the conventional Wehrl entropy. We apply a new entropic measure—the Wehrl phase distribution, defined as the phase density of the Wehrl entropy. We analyse several quantum and classical optical states of light to show the usefulness and advantages of the Wehrl phase distribution in comparison with the Wehrl entropy and conventional phase distributions, in particular those of Husimi, and Pegg and Barnett.

2. Definitions

The density matrix $\hat{\rho}$ for an arbitrary state of light can be represented by the Husimi classical-like $Q(\alpha)$-function [4]

$$Q(\alpha) = \frac{1}{\pi}\,\mathrm{Tr}(\hat{\rho}|\alpha\rangle\langle\alpha|) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \tag{1}$$

in terms of coherent states $|\alpha\rangle$. Function (1) is normalized to unity, i.e. $\int Q(\alpha)\,\mathrm{d}^2\alpha = 1$, where $\mathrm{d}^2\alpha \equiv \mathrm{dRe}\,\alpha\,\mathrm{dIm}\,\alpha = |\alpha|\,\mathrm{d}|\alpha|\,\mathrm{dArg}\,\alpha$. The Husimi representation (1) provides, equivalently to the Glauber–Sudarshan or Wigner representations, a basis for a formal equivalence between the quantum and classical descriptions of optical coherence [19].

The Wehrl classical information-theoretic entropy is defined via the Husimi Q-function as follows [3]†:

$$S_{\mathrm{W}} = -\int Q(\alpha)\ln Q(\alpha)\,\mathrm{d}^2\alpha. \tag{2}$$

The Wehrl entropy (2) is also referred to as the Shannon information of the Q-function [10]. We define the following entropic measure:

$$S_\theta = -\int Q(\alpha)\ln Q(\alpha)|\alpha|\,\mathrm{d}|\alpha| \tag{3}$$

which could be interpreted as the phase distribution‡ or phase density of the Wehrl entropy. In fact, the Wehrl entropy (2) can be obtained from (3) by simple integration,

$$S_{\mathrm{W}} \equiv -\int\!\!\int Q(\alpha)\ln Q(\alpha)|\alpha|\,\mathrm{d}|\alpha|\,\mathrm{d}\theta = \int S_\theta\,\mathrm{d}\theta \tag{4}$$

where $\theta = \mathrm{Arg}\,\alpha$. We refer to function (3) as the *Wehrl phase distribution* or *Wehrl entropy density*. We will show some formal similarities but also essential differences between the

† The minor difference between the Wehrl entropy presented here and, e.g., that applied in [8, 9, 17, 18] is the factor $\ln\pi$ resulting from the different normalization of the Husimi Q-function. However, the same normalization as ours in (1) and (3) was used by, e.g., Keitel and Wódkiewicz [11].

‡ The conventional phase distributions [21] are normalized to unity. But our definition of the Wehrl phase distribution S_θ is normalized to the state-dependent Wehrl entropy S_{W}. One can redefine S_θ as $S'_\theta = S_\theta/S_{\mathrm{W}}$ to obtain the conventional mathematical distribution. However, we keep our normalization due to clear physical interpretation of S_{W}.

Wehrl phase distribution (3) and conventional phase distributions, in particular, the so-called Husimi phase distribution [21]

$$P_\theta = \int Q(\alpha)|\alpha|\,\mathrm{d}|\alpha| \tag{5}$$

defined as the marginal function of the Husimi Q-function.

In the next sections, we will calculate the Husimi Q-function, the Wehrl entropy and the Wehrl and Husimi phase distributions for some common states of light. We will show the advantages of the entropic measure (3) over the Wehrl entropy and conventional phase distributions in a description of quantum phase properties of radiation.

3. Entropic description of fields with random phase

States with random phase are usually defined by uniform classical phase distributions [20,21], the uniform Pegg–Barnett quantum phase distribution [21, 23] or, equivalently, by rotated-quadrature distributions independent of the reference phase [22]. These definitions can also be formulated in terms of the rotationally symmetric quasiprobability distributions, in particular, the Wigner or Husimi functions [21,22]. Consequently, the states with random phase, described by the phase-independent Husimi function $Q(\alpha) = f(|\alpha|)$, also have the phase-independent entropic density S_θ. Thus, the Wehrl entropy S_W is simply related to its density S_θ by the formula $S_\theta = S_\mathrm{W}/(2\pi)$. We briefly discuss two examples of such states.

3.1. Fock states

The Fock state $|n\rangle$ is described by the phase-independent Husimi Q-function in the form of Poissonian distribution

$$Q(\alpha) = \frac{1}{\pi}\frac{|\alpha|^{2n}}{n!}\exp(-|\alpha|^2). \tag{6}$$

Consequently, the Wehrl phase distribution is given by

$$S_\theta = \frac{1}{2\pi}S_\mathrm{W} = \frac{1}{2\pi}[1 + n - n\psi(n+1) + \ln(\pi n!)] \tag{7}$$

where $\psi(n+1) = -\gamma + \sum_{k=1}^{n}\frac{1}{k}$ is digamma function defined by Euler's constant γ. Distribution (7) instantly comes from the well known expression for the Fock-state Wehrl entropy obtained, e.g., by Orłowski [17] and Peřinová *et al* [6]. The Wehrl entropy and Wehrl phase distribution for Fock states $|n\rangle$ in their dependence on photon number n are given in figures 1 and 2(a), respectively.

3.2. Chaotic field

The chaotic (Gaussian) field is defined as a state with the maximal von Neumann entropy. Its Husimi Q-function reads as [19]

$$Q(\alpha) = \frac{1}{\pi(\langle\hat{n}_\mathrm{ch}\rangle + 1)}\exp\left(-\frac{|\alpha|^2}{\langle\hat{n}_\mathrm{ch}\rangle + 1}\right) \tag{8}$$

where $\langle\hat{n}_\mathrm{ch}\rangle$ is the mean number of photons. The thermal (blackbody) radiation in thermal equilibrium at temperature T is described by (8) with the mean photon number $\langle\hat{n}_\mathrm{ch}\rangle = \{\exp(\hbar\omega/k_\mathrm{B}T) - 1\}^{-1}$, where k_B is the Boltzmann constant. The Wehrl phase distribution of the chaotic light is expressed by

$$S_\theta = \frac{1}{2\pi}S_\mathrm{W} = \frac{1}{2\pi}[\ln(\langle\hat{n}_\mathrm{ch}\rangle + 1) + \ln\pi + 1]. \tag{9}$$

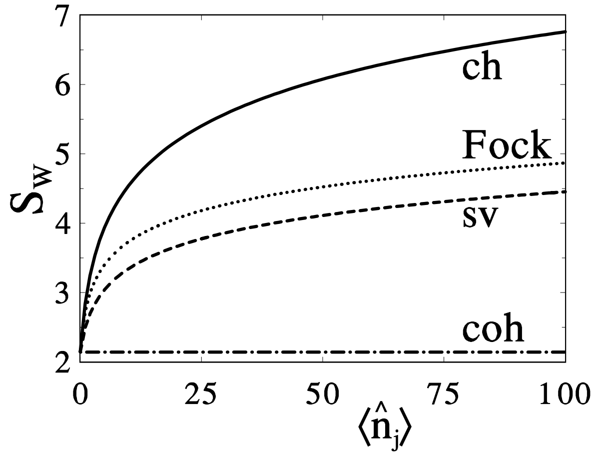

Figure 1. Wehrl entropy as a function of mean photon numbers for: Fock states ($\langle \hat{n}_{\rm Fock} \rangle = n$), chaotic field ($\langle \hat{n}_{\rm ch} \rangle$), coherent state ($\langle \hat{n}_{\rm coh} \rangle = |\alpha_0|^2$), and squeezed vacuum ($\langle \hat{n}_{\rm sv} \rangle = \sinh^2 \zeta$).

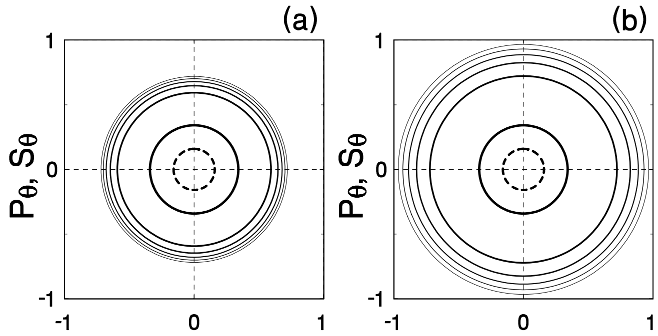

Figure 2. Wehrl phase distribution (solid circles) versus Pegg–Barnett or, equivalently, Husimi phase distribution (dashed circle) in polar coordinates for (*a*) Fock states and (*b*) chaotic fields with the mean photon numbers $n = \langle \hat{n}_{\rm ch} \rangle$ equal to: 0 (innermost solid circle), 10, 20, 30, 40, 50 (outermost circle).

The Wehrl entropy $S_{\rm W}$, given by (9), was discussed by Peřinová *et al* [6], Peřina [19], and Orłowski [17]. The Wehrl entropy for the chaotic field as a function of the mean photon number $\langle \hat{n}_{\rm ch} \rangle$ is presented in figure 1. Perfectly circular polar-coordinate representations of the phase-independent Wehrl phase distribution for the chaotic field are plotted in figure 2(*b*) for the same mean photon numbers as those depicted in figure 2(*a*) for Fock states. It is seen that Fock and chaotic fields have the same distribution S_θ in the limit of zero intensity only. With increasing intensity, the Wehrl phase distribution and, consequently, the Wehrl entropy increase faster for the chaotic field than for Fock states.

The Pegg–Barnett phase distribution [23] and the marginal quasiprobability phase distributions [21], including function (5), are equal to

$$P_\theta = \frac{1}{2\pi} \tag{10}$$

for any state with random phase. The Wehrl phase distributions (7) and (9), although independent of phase, remain dependent on the mean number of photons in the field with random phase. A comparison of the phase distributions S_θ and P_θ for the random-phase states is clearly presented in figure 2. We conclude that the Wehrl phase distribution contains more information than the conventional phase distributions, but still fulfills the requirement for a good measure of phase properties.

4. Entropic description of partial phase states and phase locking

We will show, by referring to the examples of pure or 'noisy' coherent fields, that the Wehrl phase distribution describes properly the influence of noise on the coherent field and the phase locking effect observed by the increasing mean number of photons of the partial phase states†.

4.1. Signal with noise

A superposition of coherent signal with complex amplitude α_0 and chaotic noise with the mean photon number $\langle\hat{n}_{\rm ch}\rangle$ can be described by the following Husimi Q-function [19]:

$$Q(\alpha) = \frac{1}{\pi(\langle\hat{n}_{\rm ch}\rangle + 1)} \exp\left(-\frac{|\alpha - \alpha_0|^2}{\langle\hat{n}_{\rm ch}\rangle + 1}\right). \tag{11}$$

On integration, according to definition (3), we find the closed-form expression of the Wehrl phase distribution for the superposition of coherent signal and noise as follows:

$$S_\theta = \frac{1}{2\pi} \exp[-(X_0^2 - X^2)]\{\sqrt{\pi} X[\mathrm{erf}(X) + 1] f_1 + \exp(-X^2) f_2\} \tag{12}$$

where

$$f_j = X_0^2 - X^2 + \ln[\langle\hat{n}_{\rm ch}\rangle + 1] + \ln\pi + j/2 \tag{13}$$

and

$$X = \frac{|\alpha_0|}{\sqrt{\langle\hat{n}_{\rm ch}\rangle + 1}} \cos(\theta - \theta_0) \tag{14}$$

which, for the special choice of θ, is denoted by $X_0 = X(\theta = \theta_0) = |\alpha_0|/\sqrt{\langle\hat{n}_{\rm ch}\rangle + 1}$, where θ_0 is the phase of α_0. Analogously, for the field described by (11), we derive the following Husimi phase distribution:

$$P_\theta = \frac{1}{2\pi} \exp[-(X_0^2 - X^2)]\{\sqrt{\pi} X[\mathrm{erf}(X) + 1] + \mathrm{e}^{-X^2}\} \tag{15}$$

where X is defined by (14). It is seen, by comparing (12) with (15), that the Wehrl phase distribution S_θ differs from the Husimi distribution P_θ by the factors f_j. Function (12) reduces to (15) by putting $f_1 = f_2 = 1$. The Wehrl entropy for the field (11) reads as [6, 19]

$$S_{\rm W} = \ln(\langle\hat{n}_{\rm ch}\rangle + 1) + \ln\pi + 1 \tag{16}$$

which can be obtained by direct integration of (12). The Wehrl entropy (16) is independent of the complex amplitude of the coherent signal. Distribution (12) reduces to the phase-independent function (9) for the chaotic field without coherent signal ($\alpha_0 = 0$). The Wehrl phase distribution for the Glauber coherent state is another special case of (14).

† Although, the partial phase states are usually considered to be pure (see, e.g., [23]), we analyse mixed states, which also go over into the phase states in the limit of high signal-to-noise ratio.

4.2. Coherent states

The Husimi Q-function for the Glauber coherent state $|\alpha_0\rangle$ reads as

$$Q(\alpha) = \frac{1}{\pi} \exp\{-|\alpha - \alpha_0|^2\}. \tag{17}$$

Coherent state (17) is the most common example of the partial phase state [23]. The Q-function (17) is the special case of (11) for the signal field without noise ($\langle \hat{n}_{\mathrm{ch}} \rangle = 0$). Thus, the Wehrl phase distribution for the coherent state is given by (12), while the Husimi phase distribution is equal to (15), where $f_j = X_0^2 - X^2 + \ln \pi + j/2$ and $X = |\alpha_0| \cos(\theta - \theta_0)$. The s-parametrized phase distribution for the coherent state was obtained by Tanaś *et al* [24]. Their solution in the special case for $s = -1$ reduces to the Husimi phase distribution $P_\theta(\alpha_0, n_{\mathrm{ch}} = 0)$, given by (15).

The second term in braces of (12) plays an essential role for small $|\alpha_0|$. By expanding (12) in power series of $|\alpha_0|$, and by putting $\langle \hat{n}_{\mathrm{ch}} \rangle = 0$, we find

$$S_\theta = \frac{1 + \ln \pi}{2\pi} + \frac{1 + 2 \ln \pi}{4\sqrt{\pi}} |\alpha_0| \cos(\theta - \theta_0) + \mathcal{O}(|\alpha_0|^2) \tag{18}$$

which is the approximation of the Wehrl phase distribution (12) for the coherent state with small amplitude. The corresponding Husimi phase distribution

$$P_\theta = \frac{1}{2\pi} + \frac{1}{2\sqrt{\pi}} |\alpha_0| \cos(\theta - \theta_0) + \mathcal{O}(|\alpha_0|^2) \tag{19}$$

is obtained from the exact expression (15). For $|\alpha_0| = 0$, both S_θ and P_θ are phase independent. For large $|\alpha_0|$, the first term of (12) predominates resulting in the following asymptotic formula:

$$S_\theta \approx \frac{f_1}{\sqrt{\pi}} X \exp[-(X_0^2 - X^2)]. \tag{20}$$

The error function $\mathrm{erf}(X)$ in (12) was replaced by unity in the derivation of (20). Similarly, the asymptotic Husimi phase distribution is

$$P_\theta \approx \frac{1}{\sqrt{\pi}} X \exp[-(X_0^2 - X^2)]. \tag{21}$$

The asymptotic formulae (20) and (21) are valid for $-\pi/2 \leqslant (\theta - \theta_0) \leqslant \pi/2$ only. But (12), given in terms of error function, holds for arbitrary phase θ. From (16) it follows that the Wehrl entropy for any coherent state is constant and equal to $S_{\mathrm{W}} = 1 + \ln \pi$ as shown in figure 1. In contrast, the density S_θ of the Wehrl entropy is dependent on the coherent-state intensity.

The Wehrl phase distribution is depicted for the coherent signals in figures 3(*a*)–(*c*) and for the superpositions of the coherent signal and noise in figures 3(*d*)–(*f*) for various values of the coherent-signal amplitude α_0. It is seen that, for the fixed mean number of chaotic photons, $\langle \hat{n}_{\mathrm{ch}} \rangle$, the density S_θ becomes sharper with the increasing mean number of coherent photons, $\langle \hat{n}_{\mathrm{coh}} \rangle = |\alpha_0|^2$. This is a signature of the so-called phase locking. It is known [21, 23] that the phase locking can be described (see dashed curves in figure 3) within the conventional optical phase formalisms. Here, we propose alternative entropic description of the phase locking (see the solid curves in figure 3).

Further conclusions can be drawn by comparing figures 3(*a*)–(*c*) and 3(*d*)–(*f*) for different values of the chaotic-field intensity $\langle \hat{n}_{\mathrm{ch}} \rangle$ but fixed coherent-signal intensity $|\alpha_0|^2$. We observe that (i) the area (which is the Wehrl entropy S_{W}) covered by the density S_θ increases, while (ii) the entropy density itself becomes less phase dependent with increasing noise. Thus, the Wehrl phase distribution serves as a measure of both (i) noise (phase-space uncertainty) and (ii) phase randomization (decoherence).

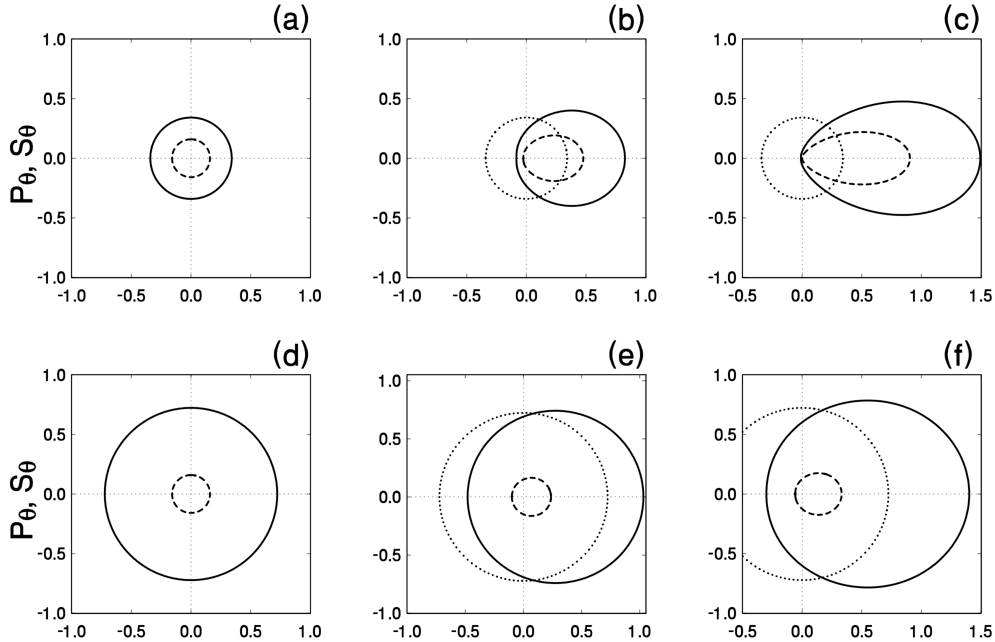

Figure 3. Wehrl (solid curves) versus Husimi (dashed curves) phase distributions in polar coordinates for coherent states (*a*)–(*c*) and superpositions of coherent signal and noise ($\langle\hat{n}_{\rm ch}\rangle = 10$) (*d*)–(*f*) with the coherent amplitude α_0 equal to: 0 (*a*), (*d*), 0.8 (*b*), (*e*), 1.6 (*c*), (*f*). Dotted circles denote the Wehrl phase distribution for $\alpha_0 = 0$.

5. Entropic description of phase bifurcation

The bifurcation phenomenon arising in the phase probability distribution of highly squeezed states was discovered by Schleich *et al* [26]. We explain this effect by using the Wehrl phase distribution.

5.1. Squeezed states

The ideal squeezed states (two-photon coherent states) are defined to be [25]

$$|\alpha_0, \zeta\rangle = \hat{D}(\alpha_0)\hat{S}(\zeta)|0\rangle \tag{22}$$

where $\hat{D}(\alpha_0) = \exp(\alpha_0\hat{a}^\dagger - \alpha_0^*\hat{a})$ is the displacement operator and

$$\hat{S}(\zeta) = \exp(\tfrac{1}{2}\zeta^*\hat{a}^2 - \tfrac{1}{2}\zeta\hat{a}^{\dagger 2}) \tag{23}$$

is the unitary squeeze operator. For simplicity, we assume that the squeeze parameter ζ is real. The squeezed vacuum is the special case of (22) for the displacement parameter $\alpha_0 = 0$. The Husimi Q-function for the state (22) is

$$Q(\alpha) = \frac{1}{\pi\sigma_1\sigma_2}\exp\left\{-\frac{\mathrm{Im}^2(\alpha-\alpha_0)}{\sigma_1^2} - \frac{\mathrm{Re}^2(\alpha-\alpha_0)}{\sigma_2^2}\right\} \tag{24}$$

where $\sigma_{1,2} = \sqrt{(\mathrm{e}^{\pm 2\zeta}+1)/2}$. We find, by assuming that α_0 is real, the following Wehrl phase distribution for the squeezed state:

$$S_\theta = \frac{1}{2\pi}\frac{\sigma_1\sigma_2}{\sigma^2}\exp[-(X_0^2 - X^2)]\{\sqrt{\pi}X[\mathrm{erf}(X)+1]f_1 + \exp(-X^2)f_2\} \tag{25}$$

where

$$f_j = \sigma_2^2 \left(\frac{X_0^2}{\sigma^2} - \frac{X^2}{\sigma_1^2} \right) + \ln(\pi\sigma_1\sigma_2) + \frac{j}{2} \tag{26}$$

and $\sigma \equiv \sigma(\theta) = \{\sigma_1^2 \cos^2\theta + \sigma_2^2 \sin^2\theta\}^{1/2}$; $X \equiv X(\theta) = \sigma_1/(\sigma_2\sigma)\alpha_0 \cos\theta$, and $X_0 = X(\theta = 0) = \alpha_0/\sigma_2$. Distribution (25) for the squeezed state has the same structure as (12) for the coherent signal field with noise, apart from the extra coefficient $\sigma_1\sigma_2\sigma^{-2}$, and modified definitions of the functions f_j and X. For $\zeta = 0$, which results in $\sigma = \sigma_1 = \sigma_2 = 1$, equation (25) reduces to a special case of (12) for the real coherent state ($\theta_0 = \langle\hat{n}_{ch}\rangle = 0$). If we put $f(i) \to 1$, function (25) will describe the Husimi phase distribution P_θ for the squeezed state obtained by Tanaś *et al* [21, 24]:

$$P_\theta = \frac{1}{2\pi} \frac{\sigma_1\sigma_2}{\sigma^2} e^{-(X_0^2 - X^2)} \{\sqrt{\pi} X[\mathrm{erf}(X) + 1] + e^{-X^2}\}. \tag{27}$$

On integrating the Wehrl phase distribution (25), one readily obtains the Wehrl entropy

$$S_W = \ln(\pi \cosh\zeta) + 1 = \tfrac{1}{2} \ln[\langle\hat{n}_{sv}\rangle + 1] + \ln\pi + 1 \tag{28}$$

in agreement with the well known results for the squeezed states shown by Lee [16], Keitel and Wódkiewicz [11] and Orłowski [17]. In (28), $\langle\hat{n}_{sv}\rangle = \sinh^2\zeta$ is the mean photon number for the squeezed vacuum. The Wehrl phase distribution (25) depends on the displacement parameter α_0 via the functions X and X_0. However, the Wehrl entropy (28) is independent of the displacement α_0.

Distribution (25) is exact and valid for arbitrary values of α_0, but has rather complicated structure in comparison with (28). We approximate the Wehrl phase distribution for a highly displaced ($\alpha_0^2 \gg 1$) squeezed state as (for $\cos\theta > 0$)

$$S_\theta \approx \frac{f_1}{\sqrt{\pi}} \frac{\sigma_1\sigma_2}{\sigma^2} X \exp[-(X_0^2 - X^2)]. \tag{29}$$

For $\alpha_0^2 \ll 1$, distribution (25) can be expanded in power series in α_0. We find the following small-amplitude Wehrl phase distribution:

$$S_\theta = \frac{1}{2\pi} \frac{\sigma_1\sigma_2}{\sigma^2} \left\{ \sqrt{\pi} X \left[\frac{1}{2} + \ln(\pi\sigma_1\sigma_2) \right] + \ln(\pi\sigma_1\sigma_2) + 1 \right\} + \mathcal{O}(\alpha_0^2). \tag{30}$$

The Wehrl phase distribution for the squeezed vacuum, i.e., in the limit $\alpha_0 \to 0$, is given by

$$S_\theta = \frac{1}{2\pi} \frac{\sigma_1\sigma_2}{\sigma^2} [\ln(\pi\sigma_1\sigma_2) + 1] \tag{31}$$

as a special case of (30). The Wehrl entropy (28) for the squeezed vacuum is presented in figure 1. Since function (28) is independent of the displacement parameter α_0, the Wehrl entropy for the arbitrary squeezed state $|\alpha_0, \zeta\rangle$ as a function of $\langle\hat{n}_{sv}\rangle = \sinh^2\zeta$ is the same as the Wehrl entropy for the squeezed vacuum plotted in figure 1. It is seen, by comparing plots in figure 1 for the same mean photon number, that the Wehrl entropy for the squeezed state is smaller than the entropies for Fock or chaotic fields, but greater than the coherent-state entropy. The Wehrl phase distribution is plotted in figures 4(*a*)–(*c*) for squeezed vacua and in figures 4(*d*)–(*f*) for squeezed states with the displacement parameter $\alpha_0 = 1$ and different values of the squeeze parameter ζ. The distribution (31) for a squeezed vacuum has a two-peak structure, which changes into a flat distribution in the limit of no squeezing ($\zeta = 0$). However, the distribution (25) for the squeezed state with nonzero displacement parameter undergoes a transition from a single- to double-peak distribution by increasing the squeeze parameter ζ or by decreasing the displacement parameter α_0. This behaviour of the entropic distribution (25) arises from the competition between the coherent component exhibiting a single-peak structure

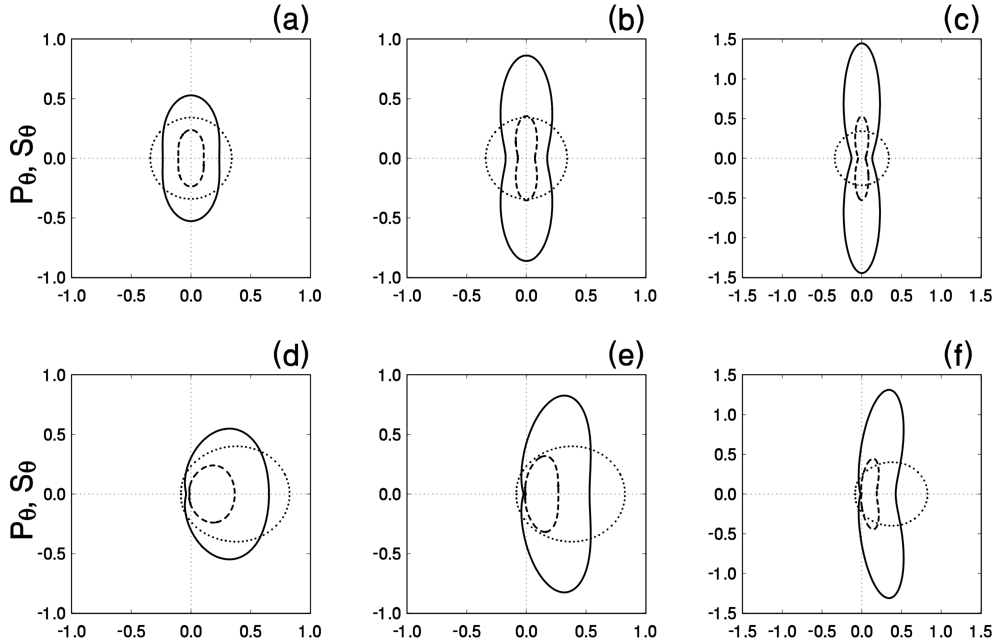

Figure 4. Bifurcation of Wehrl (solid curves) and Husimi (dashed curves) phase distributions for squeezed states with: $\alpha_0 = 0$ (squeezed vacuum) (*a*)–(*c*) and $\alpha_0 = 0.8$ (*d*)–(*f*) for different values of the squeeze parameter: $\zeta = 0.4$ (*a*), (*d*), $\zeta = 0.8$ (*b*), (*e*), and $\zeta = 1.2$ (*c*), (*f*). Dotted curves denote Wehrl phase distributions for vacuum (*a*)–(*c*) and coherent state with $\alpha_0 = 0.8$ (*d*)–(*f*).

and the squeezed-vacuum component having a double-peak structure. The above transition of the Wehrl phase distribution indicates the phase bifurcation phenomenon discovered by Schleich *et al* [26] by analysing the Pegg–Barnett phase distribution with increasing value of the product of the squeeze and displacement parameters for the state (22). Alternatively, the phase bifurcation can be analysed (see dashed curves in figure 4) in terms of the marginal quasiprobability distributions [21, 24]. The squeezed-state phase bifurcation has a simple physical interpretation according to the principle of the overlap area in the phase space [26].

In the limit of strong squeezing $\zeta \gg \alpha_0^2$, the peaks of the Wehrl phase distribution (25) are centred at $\theta = \pm\frac{\pi}{2}$ and can be approximately expressed by

$$S_\theta \approx \frac{\exp(\zeta - 2\alpha_0^2)}{2\pi}\left[\zeta + 2\alpha_0^2 + \ln\left(\frac{\pi}{2}\right) + 1\right] \tag{32}$$

which shows that the peak height of S_θ is proportional to ζe^ζ if we neglect the parameter α_0, which is small in comparison with ζ. In the same squeezing limit but for $\theta \neq \pm\frac{\pi}{2}$, distribution (25) simplifies to

$$S_\theta \approx \frac{e^{-\zeta}}{2\pi}\sec^2\theta\left\{\sqrt{2\pi}\alpha_0\left[\mathrm{erf}(\sqrt{2}\alpha_0) + \frac{\cos\theta}{|\cos\theta|}\right]f_1 + \exp(-2\alpha_0^2)f_2\right\} \tag{33}$$

where $f_j \approx \ln(\frac{\pi}{2}) + s + \frac{j}{2}$ is obtained as the approximation of (26). For squeezed vacuum, (33) reduces to

$$S_\theta \approx \frac{1}{2\pi} f_2 e^{-\zeta}\sec^2\theta. \tag{34}$$

Functions (33), (34) are proportional to $\exp(-\zeta)$, so the Wehrl phase distribution is negligible for θ not close to $\pm\frac{\pi}{2}$. The exact function (25), presented graphically in figure 4, can be approximated by (32), (33) if the condition $\zeta \gg \alpha_0^2$ is fulfilled.

The appearance of the phase bifurcation, for the fixed squeeze parameter ζ, depends on the value of the displacement parameter α_0. Since function (28) is independent of α_0, one can conclude that the conventional Wehrl entropy does not reveal the phase bifurcation effect for the squeezed state. In contrast, the Wehrl phase distribution (25), having a structure with two sharp peaks in the limit of high squeezing, properly describes this bifurcation.

6. Conclusions

We have proposed a new entropic measure—the Wehrl phase distribution, which can be interpreted as the phase density of the Wehrl classical information-theoretic entropy. We have applied the Wehrl phase distribution to describe quantum mechanical phase properties of some common optical fields, in particular, Fock states, coherent and squeezed states, and superposition of chaotic and coherent states. We have shown, in particular, that the Wehrl phase distribution clearly describes states with random phase and partial phase states, and also the effects of phase randomization, phase locking and phase bifurcation of quantum states of light. For comparison, we have also described these fields in terms of the Wehrl entropy and conventional phase distributions, including those of Husimi, and Pegg and Barnett. We have shown explicitly that the Wehrl phase distribution, in contrast to the conventional Wehrl entropy, reveals the phase bifurcation of squeezed states and phase locking of partial phase states. The Wehrl phase distribution also contains more information than the conventional phase distributions. In particular, the Wehrl phase distribution, in contrast to the Husimi or Pegg–Barnett distributions, depends on the intensity of the states with random phase (including Fock and chaotic fields). Thus, the Wehrl phase distribution combines the advantages of both the Wehrl entropy and conventional phase distributions serving as a good measure of various properties of quantum optical fields.

Acknowledgments

The authors gratefully acknowledge the helpful comments of J Bajer, M Kozierowski, M Ohya and R Tanaś. AM wishes to express his gratitude to the Japanese Ministry of Education for the scholarship. MRBW was supported by the Malaysian S&T IRPA (Grant No 09-02-03-0337). HM acknowledges support from the New Energy and Industrial Technology Development Organization (NEDO) of Japan within the Proposal-Based New Industry Creative Type Technology R&D Promotion Program.

References

[1] Ohya M and Petz D 1993 *Quantum Entropy and its Use* (Berlin: Springer)
[2] von Neumann J 1927 *Göttinger Nachr.* 273
[3] Wehrl A 1979 *Rep. Math. Phys.* **16** 353
[4] Husimi K 1940 *Proc. Phys. Math. Soc. Japan* **22** 264
see also Kano Y 1965 *J. Math. Phys.* **6** 1913
[5] Beretta G P 1984 *J. Math. Phys.* **25** 1507
[6] Peřinová V, Křepelka J, Peřina J, Lukš A and Szlachetka P 1986 *Opt. Acta* **33** 15
[7] Bužek V, Keitel C H and Knight P L 1995 *Phys. Rev.* A **51** 2575
[8] Wehrl A 1978 *Rev. Mod. Phys.* **50** 221
[9] Bužek V, Keitel C H and Knight P L 1995 *Phys. Rev.* A **51** 2594

[10] Anderson A and Halliwell J J 1993 *Phys. Rev.* D **48** 2753
[11] Keitel C H and Wódkiewicz K 1992 *Phys. Lett.* A **167** 151
[12] Vaccaro J A and Orłowski A 1995 *Phys. Rev.* A **51** 4172
[13] Watson J B, Keitel C H, Knight P L and Burnett K 1996 *Phys. Rev.* A **54** 729
[14] Orłowski A, Paul H and Kastelewicz G 1995 *Phys. Rev.* A **52** 1621
[15] Gasperini M and Giovannini M 1998 *String Theory in Curved Spacetimes* (Singapore: World Scientific) p 249
[16] Lee C T 1998 *Opt. Commun.* **66** 52
[17] Orłowski A 1993 *Phys. Rev.* A **48** 727
[18] Jex I and Orłowski A 1994 *J. Mod. Opt.* **41** 2301
[19] Peřina J 1991 *Quantum Statistics of Linear and Nonlinear Optical Phenomena* 2nd edn (Dordrecht: Kluwer)
[20] Lynch R 1995 *Phys. Rep.* **256** 367
[21] Tanaś R, Miranowicz A and Gantsog Ts 1996 *Progress in Optics* vol 35, ed E Wolf (Amsterdam: North-Holland) p 355
[22] Leonhardt U and Jex I 1994 *Phys. Rev.* A **49** R1555
[23] Pegg D T and Barnett S M 1989 *Phys. Rev.* A **39** 1665
[24] Tanaś R, Miranowicz A and Gantsog Ts 1993 *Phys. Scr.* T **48** 53
[25] Loudon R and Knight P L 1987 *J. Mod. Opt.* **34** 709
[26] Schleich W, Horowicz R J and Varro S 1989 *Phys. Rev.* A **40** 7405