

Comparative study of quantum information properties of the spin van der Waals models

A. Miranowicz,^{(a,b,c)*} Ş. K. Özdemir,^(a,b) M. Koashi,^(a,b) N. Imoto,^(a,b,d) and Y. Hirayama^(a,d)

(a) *CREST Research Team for Interacting Carrier Electronics, Japan Science and Technology Corporation (JST)*

(b) *School of Advanced Sciences, Graduate Univ. for Advanced Studies (SOKEN), Hayama, Kanagawa 240-0193, Japan*

(c) *Nonlinear Optics Division, Institute of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland*

(d) *NTT Basic Research Laboratories, 3-1 Morinosato Wakamiya, Atsugi, Kanagawa 243-0198, Japan*

Quantum information properties of the uniformly-interacting two-level quantum dots are compared for the two spin van der Waals (SVW) models with or without antiresonant excitation and deexcitation processes. Complete quantum solutions and their Schmidt decompositions are found for both models. To compare intrinsically quantum-information properties of quantum dots we study, in particular, bipartite entanglement via the von Neumann entropy. Generation of the maximally or almost maximally entangled states is analyzed in detail.

I. INTRODUCTION

Over the past few years, there has been a considerable interest, both theoretical (see [1–3] and references therein) and experimental [4,5], in the analysis of entanglement for bipartite and multipartite systems. One of the central problems is the generation of maximally entangled quantum states (MES), when the entanglement is equally shared among all (sub)systems. These states are a uniquely valuable resource for various quantum information-processing tasks. In the search of MES, the problem of a good entanglement measure becomes crucial. Arguably, the most important measure of entanglement for bipartite systems is the entanglement of formation [1]. For a bipartite pure state, say $\hat{\rho}_{AB} = (|\psi\rangle\langle\psi|)_{AB}$, the entanglement of formation is given by the von Neumann entropy defined to be $E[\hat{\rho}_{AB}] = -\text{Tr}\{\hat{\rho}_A \log_2 \hat{\rho}_A\} = -\text{Tr}\{\hat{\rho}_B \log_2 \hat{\rho}_B\}$ for the marginal density operators $\hat{\rho}_A = \text{Tr}_B\{\hat{\rho}_{AB}\}$ and $\hat{\rho}_B = \text{Tr}_A\{\hat{\rho}_{AB}\}$ of systems A and B , respectively. For a bipartite mixed state, the former definition of the entanglement of formation generalizes to the minimum average marginal entropy of ensemble decompositions of $\hat{\rho}_{AB}$. The so-called concurrence, proposed by Hill and Wootters [6] and generalized by Rungta et al. [7], is related to a pairwise entanglement of formation. Although it is “a kind of measure of entanglement in its own right” [6], the concurrence has been widely applied in analyses of various models (see, e.g., [8]–[17]). It is worth noting that a few measures have recently been proposed to describe a multipartite entanglement, but none of them satisfies all the criteria for a good entanglement measure.

The entanglement in the condensed matter systems has been studied in greater detail only very recently. Maximal nearest-neighbor (bipartite) entanglement measured in the concurrence for spin-1/2 particles was studied for their different configurations and interactions: (i) an entangled line with open ends by Wootters [8], (ii) an entangled ring by O’Connor and Wootters [9] or (iii) entangled webs by Koashi et al. [10]. Concurrence in various 1D

Heisenberg models (including those of XXX [11–13], XY [14], XXZ [15,13] models) and 1D Ising model [16] were studied under the nearest-neighbor interaction approximation. Whereas Koashi et al. [10] and Wang [17] studied concurrence in the 1D Heisenberg models under the approximation of the equivalent neighbor interactions.

In this paper we investigate the bipartite entanglement in N -dot systems described by the spin van der Waals model. We address the following question: How is the initially excited subsystem (say A) composed of M dots entangled with the remaining initially unexcited $N - M$ dots (referred to as the subsystem B)?

II. MODELS AND THEIR SOLUTIONS

We study a system of N identical two-level quantum dots constituting an insulating solid or molecular crystal. The interaction Hamiltonian is assumed to be of the Frenkel form [18]

$$\hat{H}_{\text{int}} = \hbar \sum_{n \neq m} T_{nm} \left[\hat{\sigma}_n^+ \hat{\sigma}_m^- + \hat{\sigma}_n^- \hat{\sigma}_m^+ + \gamma (\hat{\sigma}_n^+ \hat{\sigma}_m^+ + \hat{\sigma}_n^- \hat{\sigma}_m^-) \right] \quad (1)$$

for the dipole-dipole interactions among N dots; $\hat{\sigma}_n^+$ and $\hat{\sigma}_n^-$ are, respectively, the Pauli spin creation and annihilation operators for the n th dot. Terms $\hbar T_{nm}$ can describe, e.g., the transition (dynamic) dipole-dipole interactions between the n th and m th dots. Then we can put $\hbar T_{nm} = |\mathbf{r}_{nm}|^{-3} \mathbf{p}_n \cdot \mathbf{p}_m - [3\mathbf{p}_n \cdot \mathbf{r}_{nm} \mathbf{p}_m \cdot \mathbf{r}_{nm}] |\mathbf{r}_{nm}|^{-2}$, where \mathbf{p}_n is the electric dipole moment at n th site, \mathbf{r}_n is its position vector, and $\mathbf{r}_{nm} = \mathbf{r}_n - \mathbf{r}_m$. The model, given in general form (1), has no analytical solution. Thus some approximations are required in order to obtain analytical formulas. For example, we can assume the equivalent-neighbor interactions among all the dots. Then the Hamiltonian (1) simplifies to

$$\hat{H}_{\text{int}} = \kappa \sum_{n \neq m} [\hat{\sigma}_n^+ \hat{\sigma}_m^- + \hat{\sigma}_n^- \hat{\sigma}_m^+ + \gamma (\hat{\sigma}_n^+ \hat{\sigma}_m^+ + \hat{\sigma}_n^- \hat{\sigma}_m^-)], \quad (2)$$

where $\kappa = \hbar T_{nm} = \text{const}$. The model given by (2) is often referred to as the *spin van der Waals model* (see, e.g., [19]) or *Lipkin model*. It was first studied by Kittel and Shore [20] from a solid-state viewpoint and in the context of nuclear physics by Lipkin, Meshkov and Glick [21]. The equivalent-neighbor assumption can be fulfilled by choosing the symmetrical spatial configurations of N atoms. However, more realistic realization of the model for any number N of dots is based on the assumption that all the dots are coupled uniformly to an external classical field and through this field interact isotropically with one another. Various equivalent-neighbor models have been analyzed in the context of quantum-information properties in, e.g., Refs. [10,17,22,23].

The term proportional to γ corresponds to the anti-resonant excitation and deexcitation processes and can also result from the higher-order Born approximation for the atom-field interaction. This term can describe nonlinear optical phenomena including the generation of higher harmonics. Hamiltonian (2) with $\gamma = 0$ describes a quantum-dot system conserving energy (and will be referred to as the CE model), while (2) with $\gamma \neq 0$ corresponds to a non-conserving energy model (NCE model). In the following we assume $\gamma = 1$ for NCE model.

Let the initial state be described by a system of M ($M = 0, \dots, N$) dots excited and $N - M$ dots in the ground state is given as $|\psi(\gamma = 0, 0)\rangle = \{|1\rangle^M\}_A \{|0\rangle^{(N-M)}\}_B$. Then, we find the solution of the Schrödinger equation of motion for the CE SVW model is given by:

$$|\psi(\gamma = 0, t)\rangle = \sum_{m=0}^{M'} C_m^{NM}(0, t) \{|1\rangle^{M-m} |0\rangle^m\}_A \times \{|1\rangle^m |0\rangle^{N-M-m}\}_B, \quad (3)$$

where $M' = \min(M, N - M)$. The states in curly brackets, $\{|1\rangle^{(M-m)} |0\rangle^m\}$, denote a sum of all possible [i.e., $\binom{M-m}{m}$] M -dot states with the excitation number $(M - m)$. The time-dependent superposition coefficients, $C_m^{NM}(\gamma = 0, t)$, are given by

$$C_m^{NM}(\gamma = 0, t) = \sum_{n=0}^{M'} b_{mn}^{NM} \times \exp\left\{i[n(N+1-n) - M(N-M)]\kappa t\right\} \quad (4)$$

with

$$b_{mn}^{NM} = \sum_{k=0}^m (-1)^k \binom{m}{k} \binom{N-2k}{M-k}^{-1} \times \left\{ \binom{N+1-2k}{n-k} - 2 \binom{N-2k}{n-k-1} \right\}, \quad (5)$$

where $\binom{x}{y}$ are binomial coefficients. We also find that the solution of the Schrödinger equation for the NCE

model under the initial condition of $|\psi(\gamma = 1, t = 0)\rangle = |1\rangle^M |0\rangle^{N-M}$ is given by

$$|\psi(\gamma = 1, t)\rangle = \sum_{p=0}^{\nu} C_p^{(N)}(1, t) \sum_{q=0}^{\nu} \left[\theta(M - 2p) \times \{|1\rangle^{M-2p+q} |0\rangle^{2p-q}\} \{|1\rangle^q |0\rangle^{N-M-q}\} + \theta(2p - M - 1) \times \{|1\rangle^q |0\rangle^{M-q}\} \{|1\rangle^{2p+q-M} |0\rangle^{N-2p-q}\} \right], \quad (6)$$

where $\nu \equiv \text{Int}(N/2)$ is the integer part of $N/2$; by definition $|0\rangle^x = |1\rangle^x \equiv 0$ if $x < 0$; the step function $\theta(x)$ is 1 for $x \geq 0$ and 0 otherwise; the time-dependent superposition coefficients $C_p^{(N)}(\gamma = 1, t)$ are given by

$$C_p^{(N)}(\gamma = 1, t) = \sum_{r=0}^{\nu} b_{pr}^{(N)} \exp\left\{i[2r(N-r) - N(N-1)/2]\kappa t\right\} \quad (7)$$

with

$$b_{pr}^{(N)} = \sum_{k=0}^p (-1)^k \binom{p}{k} \binom{N-2k}{r-k} 2^{2k+1-N-\delta(N,2r)} \quad (8)$$

for $p = 0, 1, \dots, \nu$; $\delta(N, 2r)$ is the Kronecker delta. The upper limit p of the sum in Eq. (8) can equivalently be replaced by r or $\min(p, r)$. Similarly tight upper limits for q in Eq. (6) are equal to $\min(2p, N - M, M - 2p)$ for the first term in the sum and $\min(2p - M, N - 2p, M)$ for the second term.

The main purpose is now to apply these solutions in the analysis of the entanglement properties of the CE and NCE spin van der Waals models.

III. QUANTUM-DOT ENTANGLEMENT

The bipartite entanglement $E^{NM}(\gamma, t)$ between M dots initially excited and the remaining $(N - M)$ dots being in the ground state initially can be calculated via the Shannon entropy of the Schmidt coefficients $P_m^{NM}(\gamma, t)$ as

$$E^{NM}(\gamma, t) \equiv E[|\psi(\gamma, t)\rangle\langle\psi(\gamma, t)|] = - \sum_m P_m^{NM}(\gamma, t) \log_2 P_m^{NM}(\gamma, t). \quad (9)$$

Thus the main problem resides in the calculation of $P_m^{NM}(\gamma, t)$ from the solutions (3) and (6). We find that the Schmidt coefficients for the CE system of N dots with M dots excited initially are given in the general form as ($m = 0, \dots, M$)

$$P_m^{NM}(\gamma = 0, t) = \binom{M}{m} \binom{N-M}{m} |C_m^{NM}(0, t)|^2, \quad (10)$$

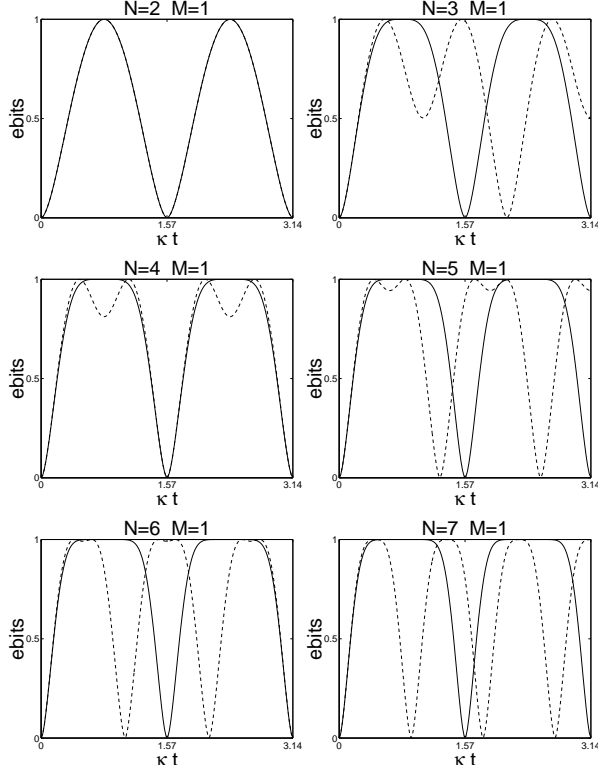


FIG. 1. Evolution of entanglement between a single dot initially excited ($M = 1$) and the remaining $N - 1 = 1, \dots, 6$ dots initially in the ground state in the NCE (solid curves) versus CE (dashed curves) models.

while for the NCE system the Schmidt coefficients are expressed in a more complex form as

$$P_m^{NM}(\gamma = 1, t) = \binom{M}{m} \sum_{n=0}^{N_m} \binom{N-M}{2n+m-M} |C_n^{NM}(1, t)|^2, \quad (11)$$

where $m = 0, \dots, M$; the upper limits of summation are $N_m = \text{Int}[\frac{N-m}{2}]$, but can simply be replaced by $\nu \equiv \text{Int}[N/2]$ recalling that $\binom{x}{y}$ vanishes for $x < y$ or $y < 0$.

Let us analyze the simplest nontrivial system of two spins when one of them is initially excited. Under interactions given by (3) and (6) the initial state $|\psi(\gamma, 0)\rangle = |10\rangle = |1\rangle_A |0\rangle_B$ evolves as ($\gamma = 0, 1$)

$$|\psi^{2,1}(\gamma, t)\rangle = C_0^{2,1}(\gamma, t) |1\rangle_A |0\rangle_B + C_1^{2,1}(\gamma, t) |0\rangle_A |1\rangle_B \\ = \cos(\kappa t) |1\rangle_A |0\rangle_B - i \sin(\kappa t) |0\rangle_A |1\rangle_B. \quad (12)$$

This is the only case when the models for $\gamma = 0$ and $\gamma = 1$ have the same solutions and thus the same entanglement evolution between subsystems A and B . The von Neumann entropy of the total system described by $\hat{\rho}^{2,1} = |\psi^{2,1}(\gamma, t)\rangle \langle \psi^{2,1}(\gamma, t)|$ is zero during the whole evolution. However, the reduced densities $\hat{\rho}_A^{2,1}$ and $\hat{\rho}_B^{2,1}$ give

nonvanishing reduced entropies $S_A = S_B = E^{2,1}(\gamma, t)$ for $\kappa t \neq k\pi/2$ ($k = 0, 1, \dots$). The bipartite entanglement evolves simply as

$$E^{2,1}(0, t) = E^{2,1}(1, t) = -\cos^2(\kappa t) \log_2 \cos^2(\kappa t) \\ - \sin^2(\kappa t) \log_2 \sin^2(\kappa t), \quad (13)$$

which is zero for $\kappa t' = k\frac{\pi}{2}$ ($k = 0, 1, \dots$) and periodically reaches its maximum of 1 ebit, i.e., $\max_t E^{2,1}(\gamma, t) = 1$. One concludes that the initial state $|10\rangle$ periodically evolves into the maximally entangled states

$$|\psi^{2,1}(\gamma, t')\rangle = \frac{|1\rangle_A |0\rangle_B + e^{-2i\kappa t'} |0\rangle_A |1\rangle_B}{\sqrt{2}} \quad (14)$$

for the evolution times $\kappa t' = (1+2k)\frac{\pi}{4}$, where $k = 0, 1, \dots$ (see Fig. 1).

In order to show explicitly the differences between the entanglement for the NCE and CE models, we analyze another simple example of a system composed of 3 dots with single initial excitation described by the state $|\psi(\gamma, 0)\rangle = |100\rangle = |1\rangle_A |00\rangle_B$. The general solution (3) for the CE model reduces to

$$|\psi^{3,1}(0, t)\rangle = C_0^{3,1}(0, t) |1\rangle_A |00\rangle_B \\ + C_1^{3,1}(0, t) |0\rangle_A \{|01\rangle\}_B, \quad (15)$$

where the superposition coefficients, given by Eq. (4), can explicitly be written as

$$C_0^{3,1}(0, t) = \frac{1}{3} [\exp(-2i\kappa t) + 2 \exp(i\kappa t)], \\ C_1^{3,1}(0, t) = \frac{1}{3} [\exp(-2i\kappa t) - \exp(i\kappa t)]. \quad (16)$$

Thus, one observes that the entanglement is given by $E^{3,1}(0, t) = -P \log_2 P - (1-P) \log_2 (1-P)$ in terms of only one Schmidt coefficient, e.g.,

$$P = P_1^{3,1}(0, t) = \frac{8}{9} \sin^2\left(\frac{3}{2}\kappa t\right). \quad (17)$$

On the other hand the solution (6) for the NCE model simplifies to

$$|\psi^{3,1}(1, t)\rangle = C_0^{3,1}(1, t) |1\rangle_A |00\rangle_B \\ + C_1^{3,1}(1, t) [|0\rangle_A \{|01\rangle\}_B + |1\rangle_A |11\rangle_B] \quad (18)$$

with the coefficients given by

$$C_0^{3,1}(1, t) = \frac{1}{4} [\exp(-3i\kappa t) + 3 \exp(i\kappa t)], \\ C_1^{3,1}(1, t) = \frac{1}{4} [\exp(-3i\kappa t) - \exp(i\kappa t)]. \quad (19)$$

Now, the entanglement $E^{3,1}(1, t)$ is given via

$$P = P_1^{3,1}(1, t) = \frac{1}{2} \sin^2(2\kappa t) \quad (20)$$

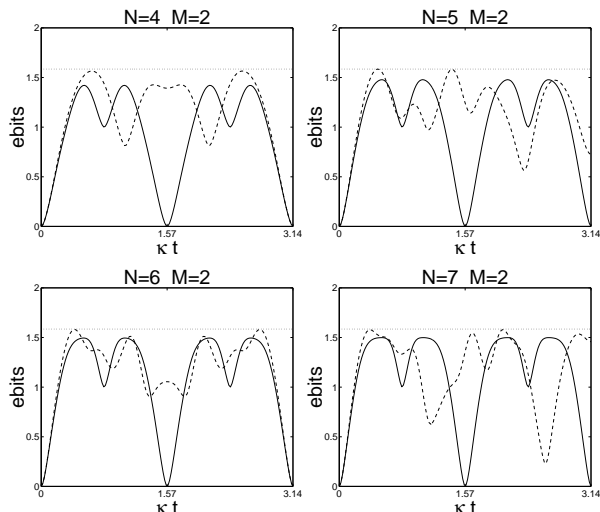


FIG. 2. As in Fig. 1 but describing the entanglement between the two dots initially excited ($M = 2$) and all the others initially unexcited. Dotted lines correspond to the MES.

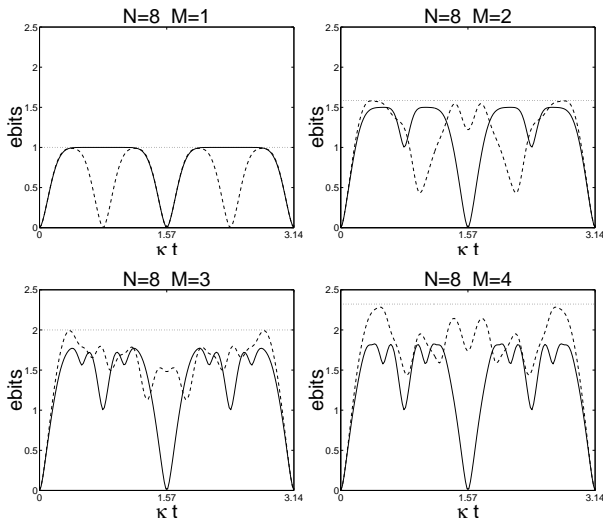


FIG. 3. Evolution of entanglement in the system of $N = 8$ dots with different excitation number M . Dotted lines correspond to the MES with $\log_2(M + 1)$ ebits of entanglement.

as also comes from (11). The Schmidt coefficients (17) and (20) differ in detail but lead to the same maximum entanglement of 1 ebit, namely

$$\max_t E^{N,1}(0, t) = \max_t E^{N,1}(1, t) = 1 \quad (21)$$

for $N = 3$. The same conclusion can be drawn for $N = 2, \dots, 6$ and $M = 1$ as seen in Fig. 1. For $N = 7$ the entanglement for the CE model is slightly less than 1 ebit, viz. $\max_t E^{7,1}(0, t) = 0.9997$. But the entanglement of 1 ebit for the NCE model can still be achieved. On the scale of Fig. 1, no differences in the maximum of entanglement even for $N = 7$ are observed.

Our solutions for the Schmidt coefficients, given by (10) and (11), enable calculation of entanglement for arbitrary N and M , although the formulas are usually (for $M \neq 1$ and $M \neq N - 1$) complicated. In Fig. 2, we compare the entanglement for systems with the initial two excitations ($M = 2$) for the NCE (solid curves) and CE (dashed curves) models. The dotted lines show the upper limit of bipartite entanglement which is observed for the maximally entangled state (MES). One can show that MES of two subsystems has d equally weighted terms in its Schmidt decomposition, giving entanglement of $\log_2(d)$ ebits, where d is the Hilbert space dimension of the smaller subsystem:

$$E_{\text{MES}}^{NM} = \log_2[\min(M, N - M) + 1]. \quad (22)$$

It is seen in Fig. 2 that the maxima of entanglement for the CE systems with $M = 2$ and $N = 4, \dots, 7$ almost reach the limit of $E_{\text{MES}}^{N,2}$. However, the maxima of entanglement for the NCE systems are much lower. In Fig. 3, we compare the dependence of entanglement on the excitation number M for a fixed number N of dots. It is seen, by increasing the excitation number M from 1 up to $\text{Int}(N/2)$, that both NCE and CE systems evolve into more and more entangled states. Nevertheless for $M > 1$, the maxima $\max_t E^{N,M}(1, t)$ for the NCE model are smaller than $\max_t E^{N,M}(0, t)$ for the CE model.

IV. CONCLUSION

We have studied evolution of quantum dots in two different spin van der Waals models by assuming (i) only resonant interactions (CE model) and by including also (ii) antiresonant excitation and deexcitation processes (NCE model). We have found solutions and their Schmidt decompositions in order to study the entanglement of the quantum dot ensembles. We have analyzed the problem how the initially excited subsystem composed of M dots can be entangled with the remaining initially unexcited $N - M$ dots. We have shown that the same maximum entanglement in the CE and NCE models can be achieved for $N = 2, \dots, 6$ dots with an initial single excitation. However for systems initially having more excitations, the maximum amount of entanglement generated in the CE model is higher than for NCE model.

ACKNOWLEDGEMENTS

We thank Jiří Bajer, Yu-Xi Liu, and Hideaki Matsueda for their stimulating discussions. This work was supported by a Grant-in-Aid for Encouragement of Young Scientists (Grant No. 12740243) and a Grant-in-Aid for Scientific Research (B) (Grant No. 12440111) by Japan Society for the Promotion of Science.

-
- * Electronic address: amirano@koryuw01.soken.ac.jp
- [1] C. H. Bennet, et al., Phys. Rev. A **53** 2046 (1996); *ibid.* **54** 3824 (1996).
- [2] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
- [3] M. Horodecki, P. Horodecki, and R. Horodecki, in: *Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments*, eds. G. Alber et al. (Springer, Berlin, 2001), quant-ph/0109124.
- [4] J. M. Raimond, M. Brune, and S. Haroche, Rev. Mod. Phys. **73**, 565 (2001).
- [5] B. Julsgaard, A. Kozhekin, and E. S. Polzik, Nature **413**, 400 (2001).
- [6] S. Hill, W. K. Wootters Phys. Rev. Lett. **78**, 5022 (1997); W. K. Wootters Phys. Rev. Lett. **80**, 2245 (1998);
- [7] P. Rungta, V. Bužek, C. M. Caves, M. Hillery, and G. J. Milburn, Phys. Rev. A **64**, 042315 (2001).
- [8] W. K. Wootters, quant-ph/0001114.
- [9] K. M. O'Connor and W. K. Wootters, quant-ph/0009041.
- [10] M. Koashi, V. Bužek, and N. Imoto, Phys. Rev. A **62**, 050302 (2000).
- [11] M.A. Nielsen, Ph. D. Thesis, Univ. of Mexico (1998), quant-ph/0011036.
- [12] M. C. Arnesen, S. Bose, and V. Vedral, quant-ph/0009060.
- [13] X. Wang, H. Fu, and A. I. Solomon, quant-ph/0105075.
- [14] X. Wang, Phys. Rev. A **64**, 012313 (2001).
- [15] X. Wang, Phys. Lett. A **281**, 101 (2001).
- [16] D. Gunlycke, S. Bose, V. M. Kendon, and V. Vedral, Phys. Rev. A **64**, 042302 (2001).
- [17] X. Wang and K. Mølmer, quant-ph/0106145.
- [18] T. Tokihiro, Y. Manabe, and E. Hanamura, Phys. Rev. A **47**, 2019 (1993).
- [19] R. Dekeyser and M. H. Lee, Phys. Rev. B **19**, 265 (1979); R. Dekeyser and M. H. Lee, Phys. Rev. B **43**, 8123 (1991); *ibid.* 8131.
- [20] C. Kittel and H. Shore, Phys. Rev. **138**, A1165 (1965).
- [21] H. J. Lipkin, N. Meshkov, and A. J. Glick, Nucl. Phys. **62**, 188 (1965); *ibid.* 199; *ibid.* 211.
- [22] L. Quiroga and N. F. Johnson, Phys. Rev. Lett. **83**, 2270 (1999); J. H. Reina, L. Quiroga, and N. F. Johnson, Phys. Rev. A **62**, 012305 (2000).
- [23] K. Mølmer and A. Sørensen, Phys. Rev. Lett. **82**, 1835 (1999); A. Sørensen and K. Mølmer, Phys. Rev. Lett. **82** (1999) 1971.

in Proc. of IEICE Conf. on
Quantum Information and Technology
 (QIT 5) (Atsugi, Japan, 2001)
 pp. 211–215