Limits of noise squeezing in Kerr effect

J. Bajer

Department of Optics, Palacký University,
17. listopadu 50, 772 00 Olomouc, Czech Republic

A. Miranowicz, R. Tanaś

Nonlinear Optics Division, Institute of Physics, Adam Mickiewicz University,
61-614 Poznań, Poland

Received 26 September 2002

It is well known that the optical Kerr effect can serve as a source of highly squeezed light, however, the analytical limit of the noise suppression has not been found yet. The process is reconsidered and an analytical estimation of the optimal quadrature noise level is found. The validity of the new scaling law is checked numerically and analytically.

PACS: 42.50.Ar, 45.65.Hw, 42.50.Dv
Key words: Kerr effect, anharmonic oscillator, squeezing, photon statistics

I Introduction

The cubic non-linearity of isotropic media, like liquid or fiber, can be responsible for the self-phase modulation, self-induced rotation of polarization, self-modulation of intensity profile or self-focusing of strong optical beam. These phenomena can simply be explained by linear dependence of the refraction index on the intensity of the propagating light, which is usually referred to as the *optical Kerr effect*. This effect plays also an important role in, e.g., the non-diffracting beam techniques, soliton propagation or non-demolition measurements. From the viewpoint of quantum optics, the Kerr effect can serve as a source of the squeezed light [1–11] (for a review see Ref. [12]), also referred to as the *self-squeezed* light. Unfortunately, the Kerr non-linearity is usually very small, thus the effective application of the Kerr effect requires long-interaction times (or lengths) and high-intensity lasers. The simplest strategy for the effective Kerr process is to use a long optical fiber and strong laser pulses. In recent experiments with the Kerr fibers, the quantum noise was successfully reduced by 0.7 dB [13] or even by 3.5 dB [14] (for a discussion of some experimental results see [3, 15]).

Quantum description of the Kerr-type evolution is among a few non-trivial quantum dynamical models, which are fully solvable. This is one of the reasons of its long-term popularity and fundamental importance. Mathematically, the Kerr process can simply be described by the well-known interaction Hamiltonian

$$\hat{H} = \tfrac{1}{2}\hbar g \hat{a}^{\dagger 2}\hat{a}^2 = \tfrac{1}{2}\hbar g \hat{n}\,(\hat{n}-1)\,, \tag{1}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are, respectively, the annihilation and creation operators satisfying the boson commutation relation; $\hbar$ is the Planck constant and g is the Kerr non-linearity. Since the photon-number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ commutes with the Hamiltonian, the photocount statistics of the field propagating through the Kerr medium is conserved. Nevertheless, the photon number fluctuations can be reduced via the Kerr effect, e.g., using the nonlinear Mach–Zehnder interferometer with the Kerr medium in one of the arms [2].

Assuming the input state to be in coherent state $|\alpha\rangle$, the solution of the Schrödinger equation is the Kerr state [12]

$$|\psi_{\mathrm{K}}\rangle = \exp\left[-\tfrac{1}{2}\mathrm{i}\tau\hat{n}(\hat{n}-1)\right]|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp\left[-\tfrac{1}{2}\mathrm{i}\tau n\,(n-1)\right]|n\rangle, \quad (2)$$

where $\tau = gL/v_{\mathrm{g}}$ is the dimensionless rescaled interaction length of the Kerr medium or the interaction time; v_{g} is the group velocity of light in the medium. The only and trivial difference between the spatial and temporal evolution is the opposite sign in front of τ. Thus, we use the terms interaction time or length interchangeably.

II Noise squeezing analysis

The Kerr dynamics has been thoroughly studied from different points of view. Many results predicted by these theoretical studies are well known. For example, it can be proved that the time evolution is exactly periodic with the period 2π. At the time moments $\tau_N = 2\pi M/N$, being rational fractions of the period, the input coherent state is transformed into a superposition of two [16] or more [17] well-localized coherent states, referred to as the Schrödinger cats and kittens, respectively. The most important result for the present study is the fact that the Kerr states evolve periodically into the quadrature noise squeezed states [1, 4–10]. By defining the general quadrature component as $\hat{X}_\theta = \hat{a}\exp(\mathrm{i}\theta) + \hat{a}^\dagger\exp(-\mathrm{i}\theta)$, where θ represents the controlled phase of a homodyne detector, the principal squeezing can be defined as the minimum of all the quadrature variances with respect to all the possible phases θ and it holds [6, 18]

$$S = \min_\theta \mathrm{Var}\hat{X}_\theta = 1 + 2\left(\langle\hat{a}^\dagger\hat{a}\rangle - |\langle\hat{a}\rangle|^2\right) - 2\left|\langle\hat{a}^2\rangle - \langle\hat{a}\rangle^2\right|. \quad (3)$$

Geometrically, the principal squeezing represents the smaller half-axis of Booth's elliptical lemniscate [19]. For vacuum and coherent fields the quadrature noise is totally independent of the choice of phase θ and it holds $S = 1$. This level of vacuum noise represents the quantum noise limit. Squeezing of the quadrature noise below the vacuum noise level occurs if $S < 1$. The formula for the principal squeezing in the Kerr effect follows from solution (2) and reads as

$$\begin{aligned} S = {} & 1 + 2|\alpha|^2\left\{1 - \exp\left[2|\alpha|^2(\cos\tau - 1)\right]\right\} \\ & - 2|\alpha|^2\left|\exp\left[|\alpha|^2\left(\mathrm{e}^{-2\mathrm{i}\tau} - 1\right) - \mathrm{i}\tau\right] - \exp\left[2|\alpha|^2\left(\mathrm{e}^{-\mathrm{i}\tau} - 1\right)\right]\right|. \end{aligned} \quad (4)$$

The time dependence of S for some amplitudes is presented in Fig. 1. As for the

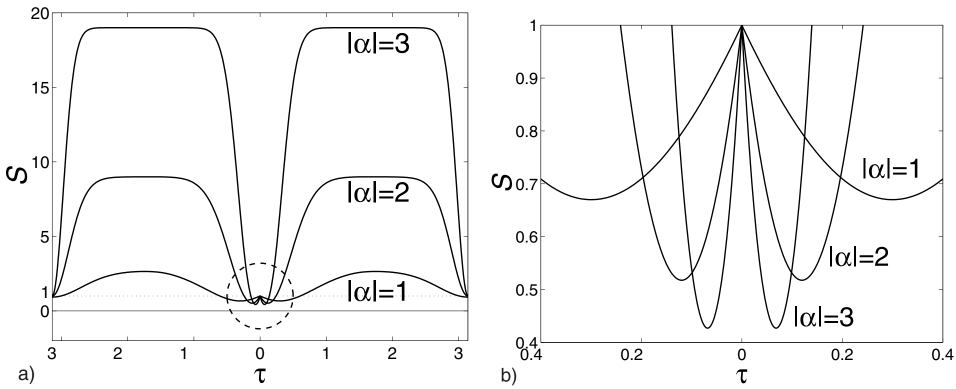

Fig. 1. Time dependence of the principal squeezing S for several values of the input amplitudes $|\alpha|$. Dotted line represents the vacuum noise level of $S = 1$. We are particularly interested in the region of the suppressed quadrature noise, which is marked in the circle in part a and presented enlarged in part b.

Kerr state, the evolution of the principal squeezing is periodic with the period of 2π. Moreover, S is an even function, therefore it is sufficient to study the evolution of τ in the interval of $(0, \pi)$ representing a half of the whole period. From Fig. 1 we can see that the principal squeezing (4) starts from the vacuum noise level $S(0) = 1$ and then evolves into the squeezed states with $S < 1$ for some limited time interval. At the moment $\tau_{\min}$ the quantum noise is maximally squeezed up to the value $S_{\min} = S(\tau_{\min})$. The phase portrait of the state has a crescent shape. The noise level can be very highly squeezed, especially for the large input amplitude $|\alpha|$, but it never reaches the absolute zero. In the following, we shall give an analytical estimation of that maximum possible noise squeezing. For larger interaction times, the quantum noise of the Kerr state goes over from the vacuum noise level with $S = 1$ up to the value of $S \approx 1 + 2|\alpha|^2$. That maximum is very flat (see Fig. 1a) and not particularly interesting for our purpose. The corresponding phase portrait rapidly changes from the crescent shape to the ring or to a discrete superposition of coherent states. At the time $\tau = \pi/2$, the Kerr state is an equally-weighted superposition of four coherent states. At last, close to $\tau \approx \pi$, the quantum noise drops down to the vacuum noise level $S \approx 1$. The phase portrait at this moment corresponds to an equally-weighted superposition of two coherent states.

The Kerr non-linearity is usually very small and, practically, the interaction times of the order of $\tau \approx 10^{-6}$ [20, 12] can be reached in optical domain. Fortunately, as follows from the above analysis and presented in Figs. 1b and 2, the highest noise suppression can be expected for short interaction times. It is worth noting that much higher Kerr-type non-linearities are observed in atom optics in, e.g., schemes based on Raman laser excitations of a trapped atom [21].

The noise power (4) depends on the interaction time τ and the input coherent amplitude $|\alpha|$. Instead of amplitude, the Kerr parameter $r = |\alpha|^2\tau$ is often used as a good measure of the Kerr interaction. For intense laser pulses, the Kerr parameter

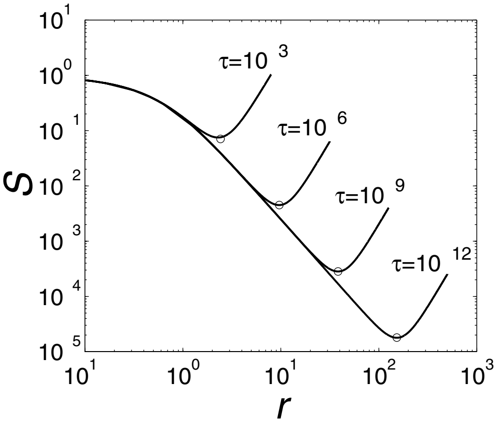

Fig. 2. Comparison of the principal squeezing minima as a function of $r = |\alpha|^2\tau$ for several values of τ. The minima apparently lie on the straight line on the log–log scale. The circles represent $S'_{\rm min}$ calculated from the scaling law formula (8). Note a good accuracy of our estimation of the minimum noise by $S'_{\rm min}$ for $\tau < 10^{-3}$.

can reach the values of the order of unity [15]. In real experiments, the interaction time is fixed by the optical fiber length, but the optical power, and thus the Kerr parameter, can simply be controlled. In the following, we shall study the principal squeezing $S\,(r)$ in its dependence on the Kerr parameter for the fixed interaction times.

From the point of view of potential applications, it is interesting to know the level of minimum noise, which can be suppressed in the Kerr process. It is tricky to find the minimum of quantum noise level analytically from the equation (4). In the former studies, it was solved only numerically and analyzed graphically. We have plotted the minima of the noise level for some values of the scaled interaction times in Fig. 2. The regular dependence of the noise minimum on the interaction time is evident. From the linear dependence of the minimum points on the logarithmic scale we can guess the scaling law of the form $S_{\rm min} \sim r_{\rm min}^{\gamma}$, which will be derived in the next section.

It is even hard to analyze numerically Eq. (4). The trouble comes from computations that require extra computer precision. For example, the curves in Fig. 2 for parameters $\tau < 10^{-6}$ cannot be computed directly from (4) using the standard 16-digit arithmetics. The correct computation requires a special computing system, like MAPLE or MATHEMATICA, that enables at least 100-digit arithmetics. On the other hand, the same numerical problems can sometimes be overcome by applying the saddle-point technique [8].

III Noise squeezing approximation

Here, as the main result of this paper, we shall derive an approximation of (4) for large amplitudes and short interaction times. This approximation can be used for simple calculation of the noise power bypassing the computer rounding error and will be used further for finding an analytic estimation of the quadrature noise minimum.

First, we exclude the amplitude from (4) by substituting $|\alpha|^2 = r/\tau$. Assuming that the interaction time τ is very short, we can expand the terms $\cos\tau - 1$, $\exp(-2\mathrm{i}\tau) - 1$ and $\exp(-\mathrm{i}\tau) - 1$ in Taylor series. To derive a useful approximation, we assume the Kerr parameter r to be finite and we sort all the terms in powers of τ. The Taylor expansion of all exponential functions in (4) leads, after final arrangement, to the approximation

$$S_1 = 1 + 2r^2 - 2r\sqrt{1+r^2} - r^3\tau + \frac{(3r^2+5)\,r^2\tau}{\sqrt{1+r^2}}, \tag{5}$$

which is linear in τ. The accuracy of the approximation (5) in comparison with the exact values of squeezing for two different values of τ can be seen in Fig. 3. In the less accurate approximation, the linear correction in (5) can be ignored. In that case we get the well-known approximation [8, 15]

$$S_0 = 1 + 2r^2 - 2r\sqrt{1+r^2}, \tag{6}$$

which is indeed not sufficient to determine the minimum noise level. Equation (6) only depends on the Kerr parameter r and thus gives physical grounds for the introduction of r. As shown in Fig. 3, the S_0 approximation is monotonic in r and has the only minimum of $S_0 = 0$ in the limit of $r \to \infty$. For sufficiently large r, S_0 can be estimated as $S_0 \approx 1/(4r^2)$. On the other hand, for the relatively small Kerr parameters, (6) can be approximated linearly as $S_0 \approx 1 - 2r$. The vast majority of the up-to-date measurements have been performed in the domain where the approximation (6) is valid.

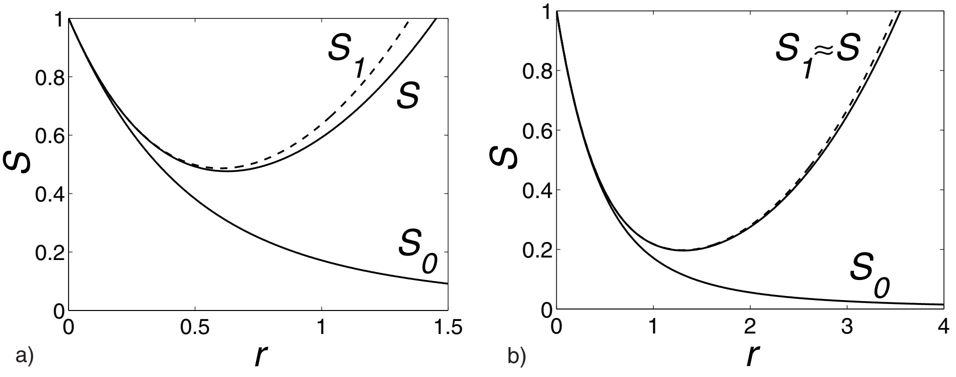

Fig. 3. Comparison of the approximations S_0 and S_1, and the exact values S of the principal squeezing as a function of the Kerr parameter r for a) $\tau = 10^{-1}$ and b) $\tau = 10^{-2}$. The figure shows that, for $\tau \leq 10^{-2}$, the approximation S_1 can be used to estimate the minimum quadrature noise with a good precision.

Table 1. Several numerical values of the principal squeezing minimum, $S_{\min}$, and the corresponding Kerr parameter $r_{\min}$ obtained at the scaled interaction times τ. Their estimations $S'_{\min}$ and $r'_{\min}$ are based on the scaling law approximation given by (8). It can be concluded from the table that (i) the level of the minimal quadrature noise regularly drops down with the shorter interaction time τ, which, however, requires higher intensity such that the Kerr parameter $r = |\alpha|^2\tau$ increases, and (ii) the estimation is valid for $\tau < 10^{-3}$.

τ	$S_{\min}$	$S'_{\min}$	$r_{\min}$	$r'_{\min}$
10^{-1}	0. 476	0. 448	0. 627	0. 964
10^{-2}	0. 196	0. 178	1. 312	1. 528
10^{-3}	7.503×10^{-2}	7.103×10^{-2}	2. 288	2. 421
10^{-4}	2.899×10^{-2}	2.828×10^{-2}	3. 755	3. 839
10^{-6}	4.501×10^{-3}	4.482×10^{-3}	9. 609	9. 642
10^{-9}	2.829×10^{-4}	2.828×10^{-4}	38. 38	38. 38
10^{-12}	1.784×10^{-5}	1.784×10^{-5}	152. 8	152. 8

By assuming that the Kerr parameter of the minimum searched is large (see the values in Table 1), we can simplify (5) even more. We expand (5) in the asymptotic series for large r and keep the largest two terms only. Thus, we obtain

$$S_1 \approx S' = \frac{1}{4r^2} + 2r^3\tau. \tag{7}$$

From this elementary approximation we directly obtain a formula for the minimum of the principal squeezing

$$S'_{\min} = \tfrac{5}{12}\,(12\tau)^{2/5} \quad \text{at} \quad r'_{\min} = (12\tau)^{-1/5}\,, \tag{8}$$

leading to $S'_{\min} = 5/(12r'^2_{\min})$. This estimation of the minimum noise level is an analytical confirmation of the scaling law as observed in Fig. 2. We have shown that the minimum level of quadrature noise is proportional to the 2/5th power of the interaction time or the fiber length. The shorter interaction time $\tau \pmod{2\pi}$ results in the deeper reduction of noise. But this requires the higher optical power to reach the minimum since it holds $|\alpha|^2 = r_{\min}/\tau \sim \tau^{-6/5}$. For example, to obtain $S_{\min} \approx 4.5 \times 10^{-3}$ at the interaction time $\tau \approx 10^{-6}$, the optimum values of the Kerr parameter and the intensity are $r_{\min} \approx 9.6$ and $|\alpha|^2 \approx 10^7$, respectively. A comparison of the exact values S and τ, calculated from (4), with their estimations S' and τ', given by the scaling law (8), is presented in Table 1 and Fig. 2, where the estimations are marked by small circles.

IV Conclusion

We have analyzed ordinary coherent light interacting with a non-absorbing nonlinear Kerr medium, modelled as an anharmonic oscillator. Former numerical studies have shown that the model leads to almost complete quadrature squeezing. We have found, as we believe for the first time, an approximate analytical formula for the optimal noise level for high intensity coherent light, assuming realistic values (see, e.g., [20]) of the Kerr nonlinearity.

JB was supported by the EU grant under the QIPC project IST-1999-13071 (QUICOV) and the Czech Ministry of Education within the project LN00A015. AM was supported by the Japan Science Technology Corporation (JST-CREST).

References

[1] R. Tanaś and S. Kielich: Opt. Commun. **45** (1983) 351.
R. Tanaś: *Coherence and Quantum Optics V* (Eds. L. Mandel and E. Wolf), Plenum, New York, 1984, p. 645.

[2] Y. Yamamoto, N. Imoto, and S. Machida: Phys. Rev. A **33** (1986) 3243.
M. Kitagawa and Y. Yamamoto: Phys. Rev. A **34** (1986) 3974.

[3] G.J. Milburn, M.D. Levenson, R.M. Shelby, S.H. Perlmutter, R.G. DeVoe, and D.F. Walls: J. Opt. Soc. Am. B **4** (1987) 1476.

[4] G.J. Milburn: Phys. Rev. A **33** (1986) 674.

[5] C.C. Gerry: Phys. Rev. A **35** (1987) 2146.

[6] A. Lukš, V. Peřinová, and Z. Hradil: Acta Phys. Polon. A **74** (1988) 713.

[7] V. Bužek: Phys. Lett. **136** (1989) 188.

[8] R. Tanaś, A. Miranowicz, and S. Kielich: Phys. Rev. A **43** (1991) 4014.

[9] H.X. Meng, C.L. Chai, and Z.M. Zhang: Phys. Rev. A **48** (1993) 3219.

[10] K. Sundar: Phys. Rev. A **53** (1996) 1096.

[11] J.H. Shapiro and A. Shakeel: J. Opt. Soc. Am. B **14** (1997) 232.

[12] R. Tanaś: *Theory of Non-Classical States of Light* (Eds. V. Dodonov and V.I. Man'ko), Taylor & Francis, London, 2002.

[13] M.D. Levenson, R.M. Shelby, and S.H. Perlmutter: Opt. Lett. **10** (1985) 514.

[14] K. Bergman and H.A. Haus: Opt. Lett. **16** (1991) 663.
H.A. Haus: J. Opt. Soc. Am. B **12** (1995) 2019.

[15] H.A. Bachor: *A Guide to Experiments in Quantum Optics*, Wiley-VCH, Weinheim, 1998, p. 267.

[16] B. Yurke and D. Stoler: Phys. Rev. Lett. **57** (1986) 13.
A. Mecozzi and P. Tombesi: Phys. Rev. Lett. **58** (1987) 1055.

[17] A. Miranowicz, R. Tanaś, and S. Kielich: Quantum Opt. **2** (1990) 253.

[18] A. Lukš, V. Peřinová, and J. Peřina: Opt. Commun. **67** (1988) 149.

[19] R. Loudon: Opt. Commun. **70** (1989) 109.

[20] H.H. Ritze and A. Bandilla: Opt. Commun. **30** (1979) 125.

[21] S. Wallentowitz and W. Vogel: Phys. Rev. A **55** (1997) 4438.
J.K. Breslin, C.A. Holmes, and J. Milburn: Phys. Rev. A **56** (1997) 3022.