Classical information entropy for single and two mode quantum fields

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ABSTRACT

The classical information entropy defined by Wehrl in terms of the Husimi Q-function is discussed and generalized over the concepts of the Wehrl phase distribution[1,2] and the Wehrl intermode-correlation parameters.[3] The classical entropic functions are applied to describe the quantum properties of single and/or two-mode optical fields.

Keywords: Wehrl entropy, quantum entropy, intermode correlations, phase space

1. INTRODUCTION

The entanglement as well the quantum information entropy are the most striking and interesting concepts in quantum mechanics. Quantum entropy, proposed by von Neumann,[4] as a natural generalization of the Boltzmann classical entropy can be applied for many quantum-mechanical problems. For instance, it can be useful as a measure of quantum entanglement, quantum optical correlations, photocount statistics, quantum decoherence and noise, purity of states and many others. The von Neumann entropy becomes zero for all pure states, and hence cannot be used for discriminating them, whereas paradoxically, the Wehrl classical entropy is useful for this purpose. In this paper we show how quantities defined on the basis of the Wehrl entropy can be applied for the discrimination of the single-mode quantum fields and, in addition, to describe phase-space correlations of the two-mode fields.

2. SINGLE-MODE FIELDS

The Wehrl entropy definition is based on the Husimi $Q(\alpha)$-function corresponding to a given quantum field[5]

$$Q(\alpha) = \frac{1}{\pi}\mathrm{Tr}\,(\hat{\rho}|\alpha\rangle\langle\alpha|) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \ , \tag{1}$$

where $\hat{\rho}$ is the density matrix for the quantum field, whereas $|\alpha\rangle$ denotes a coherent state. Obviously, the Q-function is normalized, i.e.

$$\int Q(\alpha)d^2\alpha = 1 \ , \tag{2}$$

where $d^2\alpha = d\mathrm{Re}\,\alpha\,d\mathrm{Im}\,\alpha = |\alpha|d|\alpha|\,dArg\alpha$, and is referred to as *quasi-probability* function. The Wehrl classical information-theoretic entropy is defined via this function as follows[6]

$$S_W = \int Q(\alpha)\ln Q(\alpha)\,d^2\alpha \ . \tag{3}$$

This entropy is also referred to as the Shannon information of the Q-function.

To discriminate various quantum fields we can introduce other quantities, for instance the *Wehrl phase distribution* or *Wehrl entropy density*[1] defined as:

$$S_\Theta = \int Q(\alpha)\ln Q(\alpha)\,|\alpha|\,d|\alpha| \ , \tag{4}$$

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which is simply related to the Wehrl entropy by integration:

$$S_W = \int \int Q(\alpha) \ln Q(\alpha) \, |\alpha| \, d|\alpha| d\Theta = \int S_\Theta d\Theta \quad , \tag{5}$$

where $\Theta = \mathrm{Arg}\,\alpha$. S_Θ can be interpreted as a phase distribution generalizing the so-called Husimi phase distribution

$$P_\Theta = \int \int Q(\alpha) \, |\alpha| \, d|\alpha| \tag{6}$$

defined as the marginal function of the Husimi Q-function.

To examine how useful the entropic functions are, we can calculate appropriate quantities for various quantum optical fields. One of these states are the n-photon *Fock* states. The Q-function corresponding to the Fock states can be written as

$$Q(\alpha) = \frac{1}{\pi} \frac{|\alpha|^{2n}}{n!} \exp\left(-|\alpha|^2\right) \quad , \tag{7}$$

which is in the form of Poissonian distribution. As a consequence, the Wehrl distribution S_Θ can be expressed as[7,8]

$$S_\Theta = \frac{1}{2\pi} S_W = \frac{1}{2\pi}\left[1 + n - n\psi(n+1) + \ln(\pi n!)\right] \quad , \tag{8}$$

where $\psi(n+1) = -\gamma + \sum_{k=1}^{n}$ is digamma function defined using the Euler γ constant.

Another example of the optical field is chaotic field. For this case the Husimi Q-function can be expressed as[9]

$$Q(\alpha) = \frac{1}{\pi(\langle \hat{n}_{ch}\rangle + 1)} \exp\left(-\frac{|\alpha|^2}{\langle \hat{n}_{ch}\rangle + 1}\right) \quad , \tag{9}$$

where $\langle \hat{n}_{ch}\rangle$ is the mean number of photons corresponding to the black-body thermal radiation at temperature T and is equal to

$$\langle \hat{n}_{ch}\rangle = \frac{1}{\exp\frac{\hbar\omega}{k_B T} - 1} \quad . \tag{10}$$

The quantity k_B appearing here is the Boltzmann constant. As a consequence, the Wehrl distribution S_Θ for the chaotic field is given by[7-9]

$$S_\Theta = \frac{1}{2\pi} S_W = \frac{1}{2\pi}\left[\ln(\langle \hat{n}_{ch}\rangle + 1) + \ln \pi + 1\right] \quad . \tag{11}$$

Since, both fields discussed here are of the random phase, the Wehrl distributions are phase independent and are equal to the Wehrl entropies after multiplication by the factor 2π. However, those quantities depend in various ways on the mean number of photons. Fig. 1 shows the Wehrl entropies as a function of mean number of photons for phase-independent (such as Fock states and chaotic fields), but also phase-dependent fields including squeezed vacuum, and Glauber and two-photon coherent states. It is visible that the dependencies of S_W on $\langle \hat{n}\rangle$ are strongly determined by the character of the quantum field.

The Pegg-Barnett phase distribution[10] and the marginal quasiprobability phase distributions[11] are equal to $1/(2\pi)$ for any state with random phase, as depicted by broken circle in Figs. 2(d) and 2(e). The corresponding Wehrl phase distributions S_Θ are phase independent too. However, they depend on the mean number of photons. Figs. 2(d,e) show a comparison of the phase distributions S_Θ and P_Θ for the states with random phase. It is seen that the Wehrl distributions are more informative then the conventional phase distributions (including the Husimi (6) or Pegg-Barnett phase distributions). On the other hand, curves in Figs. 2(b,c) corresponding to a single curve in Fig. 1, show that the Wehrl phase distribution is also more informative than the Wehrl entropy.
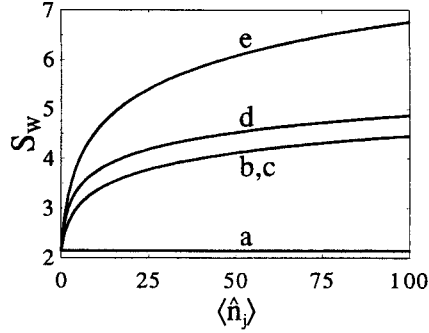
Figure 1. Wehrl entropy as a function of mean number of photons for various single-mode fields: (a) coherent state ($\langle \hat{n}_{coh} \rangle = |\alpha|^2$); (b) squeezed vacuum and (c) two-photon coherent state (squeezed state) for $\langle \hat{n}_{sv} \rangle = \sinh^2 \xi$; (d) Fock states ($\langle \hat{n}_{Fock} \rangle = n$), and (e) chaotic field ($\langle \hat{n}_{ch} \rangle$).
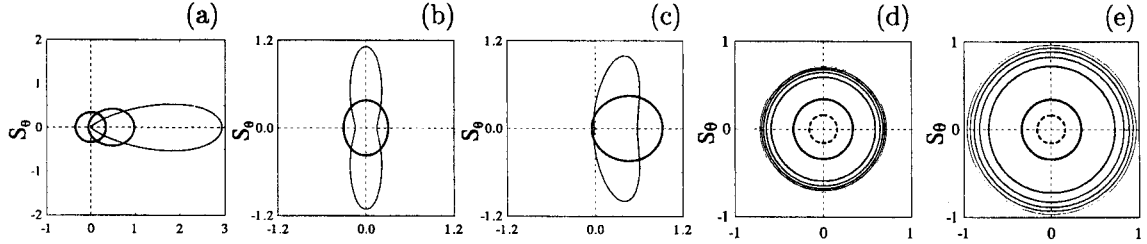
Figure 2. Wehrl phase distributions for the same fields as in Fig. 1: (a) coherent states for $\alpha = 0, 1, \sqrt{10}$; (b) squeezed vacuum for $\xi = 0.1, 1$; (c) two-photon coherent states for $\alpha = 1$ and $\xi = 0.1, 1$; (d) Fock states and (e) chaotic fields with $n = \langle \hat{n}_{ch} \rangle = 0, 10, \cdots, 50$. Curves from the thickest to the thinnest correspond to increasing parameters. Additionally, broken circles in figures (d) and (e) correspond to the Pegg-Barnett phase distributions being the same for all values of $\langle \hat{n}_{ch} \rangle$ and n.

3. TWO-MODE FIELDS

The intermode correlations, related to the entanglement problems, seem to be one of the most interesting subjects of quantum information theory. As a criterion of the pure-state bipartite entanglement, the von Neumann entropy (quantum entropy) of one subsystem has been already used.[12,13] Since, in this paper we analyze nonclassical correlations between modes in phase-space for the two-mode squeezed vacuum, we will apply the Wehrl entropy concept. For this case we apply the definitions of the parameters describing degree of the intermode correlations.[3] Thus, for the two-mode systems defined in the Hilbert space $H = H_A \otimes H_B$ we introduce the following quasiprobability $Q(\alpha_1, \alpha_2; \beta_1, \beta_2)$:

$$Q(\alpha_1, \alpha_2; \beta_1, \beta_2) = \frac{1}{\pi} \langle \alpha\beta | \hat{\rho}_{AB} | \alpha\beta \rangle \ , \tag{12}$$

where $|\alpha\beta\rangle = |\alpha\rangle \otimes |\beta\rangle$ and $\hat{\rho}_{AB}$ is the density operator defined in H. The Wehrl entropy for such a system can be written as:

$$S[\alpha_1, \alpha_2; \beta_1, \beta_2] = - \int d\alpha_1 d\alpha_2 \, d\beta_1 \, d\beta_2 \, J \, Q(\alpha_1, \alpha_2; \beta_1, \beta_2) \ln Q(\alpha_1, \alpha_2; \beta_1, \beta_2) \ , \tag{13}$$

with $J = 1$ if $(\alpha_1, \alpha_2) + (\alpha_x, \alpha_y)$, or $J = \alpha_r$ if $(\alpha_1, \alpha_2) + (\alpha_r, \alpha_\phi)$. The $Q(\alpha_1, \alpha_2; \beta_1, \beta_2)$ function depends on four variables of phase space. Hence, it allows us to define quasiprobabilities that represent individual modes:

$$Q(\alpha_1, \alpha_2) = \int d\beta_1 \, d\beta_2 \, J \, Q(\alpha_1, \alpha_2; \beta_1, \beta_2)$$

$$Q(\beta_1, \beta_2) = \int d\alpha_1 \, d\alpha_2 \, J \, Q(\alpha_1, \alpha_2; \beta_1, \beta_2) \ , \tag{14}$$

and the quasiprobabilities referred to as intermode distributions:

$$Q(\alpha_i, \beta_j] = \int d\alpha_p \, d\beta_q \, J \, Q(\alpha_1, \alpha_2; \beta_1, \beta_2) \ , \tag{15}$$

with $i, j, p, q = 1, 2$ and $i \neq p, j \neq q$. Those quasiprobabilities can be used in the definitions of the following entropies

$$
\begin{aligned}
S[\alpha_1, \alpha_2] &= -\int d\alpha_1 d\alpha_2 \, J \, Q(\alpha_1, \alpha_2) \ln Q(\alpha_1, \alpha_2) \\
S[\alpha_i, \beta_j] &= -\int d\alpha_i d\beta_j \, J \, Q(\alpha_i; \beta_j) \ln Q(\alpha_i, \beta_j) \\
S[\alpha_i] &= -\int d\alpha_i \, J \, Q(\alpha_i) \ln Q(\alpha_i)
\end{aligned}
\tag{16}
$$

$$\tag{17}$$

fulfilling the nonadditivity relation:

$$S\left[\sum_i U_i\right] \leq \sum_i S_i[U_i] \quad \text{for} \quad U = \sum_i U_i \tag{18}$$

To examine the correlations in phase-space we introduce the quantity

$$I[u, v] = S[u] + S[v] - S[u, v] \ , \tag{19}$$

where $I[u, v]$ measures the information contained in the variable u about the variable v, and the mutual information concept has been applied.[3] Moreover, it is possible to examine other quantities measuring correlations in complex systems. Another example can be the parameter L defined as[3]

$$
\begin{aligned}
L' &= S[\alpha_1; \alpha_2] + S[\beta_1; \beta_2] - (S[\alpha_1; \beta_1] + S[\alpha_2; \beta_2]) \\
L'' &= S[\alpha_1; \alpha_2] + S[\beta_1; \beta_2] - (S[\alpha_1; \beta_2] + S[\alpha_2; \beta_1])
\end{aligned}
\tag{20}
$$

and is related to I by the relations

$$
\begin{aligned}
L' &= I[\alpha_1; \beta_1] + I[\alpha_2; \beta_2] - (I[\alpha_1; \alpha_2] + S[\beta_1; \beta_2]) \\
L'' &= I[\alpha_1; \beta_2] + I[\alpha_2; \beta_1] - (I[\alpha_1; \alpha_2] + S[\beta_1; \beta_2]) \ .
\end{aligned}
\tag{21}
$$

For any uncorrelated systems these parameters are non-positive. Otherwise, when correlations occur in the system, both parameters (or one of them) are positive i.e. $l' > 0$ and(or) $l'' > 0$.

As an example of application of the above parameters we shall discuss the properties of the two-mode squeezed vacuum state $|\xi\rangle$

$$|\xi\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} (\tanh r)^n e^{in\psi} |n\rangle_a |n\rangle_b \ . \tag{22}$$

The quasiprobability Q for this state takes the following form

$$Q_\xi(a, \phi_a; b, \phi_b) = \frac{1}{\pi^2 \cosh^2 r} e^{-a^2 - b^2} \sum_{n,m=0}^{\infty} \frac{(ab \tanh r)^{n+m}}{n!m!} e^{-i(\phi_a + \phi_b - \psi)(n-m)} \ , \tag{23}$$

where the coordinates in the phase-space corresponding to the modes a and b are specified as follows

$$\alpha = a \, e^{i\phi_a} \quad \text{and} \quad \beta = b \, e^{i\phi_b} \ , \tag{24}$$

defining the amplitude and phase of each mode. The Q-function (23) enables us to calculate the parameters measuring correlations in the system. For the state discussed here these parameters are:

$$I[a, \phi_a] = 0, \qquad I[b, \phi_b] = 0,$$
$$I[a, \phi_b] = 0, \qquad I[b, \phi_a] = 0. \tag{25}$$

This result means that we have no correlations between the amplitude and the phase for the single mode. Moreover, there are no correlations between the amplitude and phase of different modes. Contrary, the mutual information for phases of two modes differs from (25), is positive and equal to

$$I[\phi_a; \phi_b] = 2 \ln 2\pi - \int_0^{2\pi} d\phi_a d\phi_b Q_\xi(\phi_a; \phi_b) \ln Q_\xi(\phi_a; \phi_b) . \tag{26}$$

Similarly, the form of the appropriate formula for the mutual information of the amplitudes points that the correlations between two amplitudes of the modes exist and depend on the value of the squeezing parameter r:

$$I[a; b] = \ln \cosh^2 r - 2 \tanh^2 r + \int_0^\infty da \, db \, a \, b \, Q_\xi(a; b) \ln B_0(ab \tanh r) . \tag{27}$$

The quantity B_0 appearing in(27) is a modified Bessel function.

We see, from the examples discussed in this communication, that the Wehrl entropy based parameters can enable us to investigate various quantum mechanical properties of quantum fields. Those quantities can help distinguish various fields and investigate their properties from the point of view of the quantum information theory. This fact concerns not only single-mode fields but two– and multi-modes fields too. For instance, we are able to investigate their internal correlations and other properties related to the information theory.

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