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A comparative study of relative entropy of entanglement, concurrence and negativity

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Abstract
The problem of ordering of two-qubit states imposed by the relative entropy of entanglement (E) in comparison with the concurrence (C) and negativity (N) is studied. Analytical examples of states consistently and inconsistently ordered by the entanglement measures are given. In particular, the states for which any of the three measures imposes order opposite to that given by the other two measures are described. Moreover, examples are given of pairs of the states for which (i) $N' = N''$ and $C' = C''$ but E' is different from E'', (ii) $N' = N''$ and $E' = E''$ but C' differs from C'', (iii) $E' = E''$, $N' < N''$ and $C' > C''$; and (iv) states having the same E, C and N but still violating the Bell–Clauser–Horne–Shimony–Holt inequality to different degrees.

Keywords: quantum entanglement, relative entropy, negativity, concurrence, Bell inequality

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Quantum entanglement is a key resource for quantum information processing, but its mathematical description is still far from completion [1] and its properties are more and more intriguing. In particular, Eisert and Plenio [2], five years ago, observed from Monte Carlo simulation of pairs of two-qubit states σ' and σ'' that entanglement measures (say $E^{(1)}$ and $E^{(2)}$) do not necessarily imply the same ordering of states. This means that the intuitive requirement

$$E^{(1)}(\sigma') < E^{(1)}(\sigma'') \Leftrightarrow E^{(2)}(\sigma') < E^{(2)}(\sigma'') \qquad (1)$$

can be violated. The problem was then analysed by others [3–11]. In particular, Virmani and Plenio [4] proved that all good asymptotic entanglement measures are either identical or fail to impose consistent orderings on the set of all quantum states. Here, an entanglement measure is referred to as 'good' if it satisfies (at least most of) the standard criteria [12–14] including the criterion that for pure states it should reduce to the canonical form given by the von Neumann entropy of the reduced density matrix.

We will study analytically the problem of ordering of two-qubit states imposed by the following three standard entanglement measures.

The first measure to be analysed here is the relative entropy of entanglement (REE) of a given state σ, which is defined by Vedral *et al* [12, 13] (for a review see [15]) as the minimum of the quantum relative entropy $S(\sigma \parallel \rho) = \mathrm{Tr}(\sigma \lg \sigma - \sigma \lg \rho)$ taken over the set $\mathcal{D}$ of all separable states ρ, namely

$$E(\sigma) = \min_{\rho \in \mathcal{D}} S(\sigma \parallel \rho) = S(\sigma \parallel \bar{\rho}), \qquad (2)$$

where $\bar{\rho}$ denotes a separable state closest to σ. We assume, for consistency with the other entanglement measures, that lg stands for $\log_2$ although in the original Vedral *et al* papers [12, 13] the natural logarithms were chosen. It is usually difficult to calculate the REE analytically with the exception of the case for states with high symmetry, including those discussed in sections 3 and 4. Thus, in general, the REE is calculated numerically using the methods described in, e.g., [13, 16, 17]. The REE satisfies both continuity and convexity (monotonicity under discarding information, $E(\sum_i p_i\sigma_i) \leqslant \sum_i p_i E(\sigma_i)$) [18], but it does not fulfil additivity ($E(\sigma_1 \otimes \sigma_2) = E(\sigma_1) + E(\sigma_2)$) [19].

The second measure of entanglement for a given two-qubit state σ is the Wootters concurrence $C(\sigma)$ defined as [20]

$$C(\sigma) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \qquad (3)$$

where the λ_is are the square roots of the eigenvalues of $\sigma(\sigma^{(y)} \otimes \sigma^{(y)})\sigma^*(\sigma^{(y)} \otimes \sigma^{(y)})$ put in nonincreasing order, $\sigma^{(y)}$

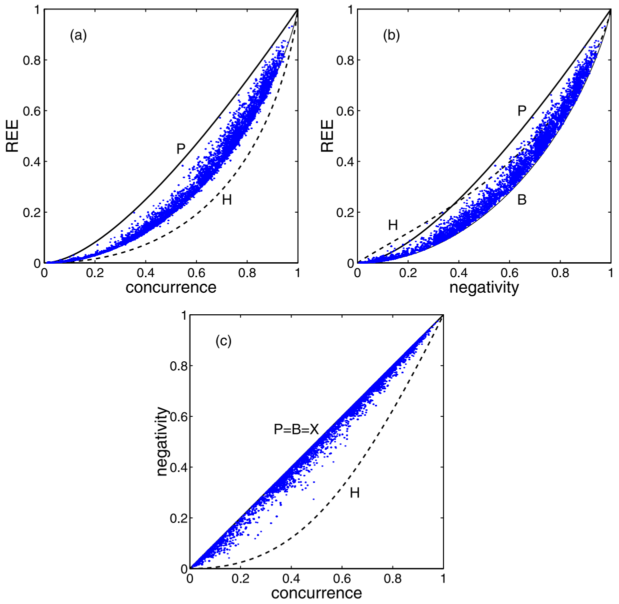

Figure 1. Numerical simulations of about 10^5 quantum states σ: (a) REE $E(\sigma)$ versus concurrence $C(\sigma)$, (b) $E(\sigma)$ versus negativity $N(\sigma)$ and (c) $N(\sigma)$ versus $C(\sigma)$. Curves correspond to the Horodecki (H), pure (P), Bell diagonal (B) and σ_X (X) states.

is the Pauli spin matrix and the asterisk stands for complex conjugation. The concurrence $C(\sigma)$ is monotonically related to the entanglement of formation $E_{\mathrm{form}}(\sigma)$ [21] as given by the Wootters formula [20]

$$E_{\mathrm{form}}(\sigma) = h\left(\tfrac{1}{2}\left[1+\sqrt{1-C^2(\sigma)}\right]\right) \tag{4}$$

in terms of the binary entropy $h(x) = -x\lg x - (1-x)\lg(1-x)$. The concurrence and entanglement of formation satisfy convexity [20, 22]. But, to our knowledge, the question of additivity of the entanglement of formation is still open [22, 23].

The third useful measure of entanglement is the negativity—a measure related to the Peres–Horodecki criterion [24] as defined by

$$N(\sigma) = 2\sum_j \max(0, -\mu_j), \tag{5}$$

where μ_js are the eigenvalues of the partial transpose σ^Γ of the density matrix σ of the system. Note that for any two-qubit states, σ^Γ has at most one negative eigenvalue. As shown by Audenaert *et al* [25] and as a subsidiary by Ishizaka [26], the negativity of any two-qubit state σ is a measure closely related to the PPT entanglement cost as follows:

$$E_{\mathrm{PPT}}(\sigma) = \lg[N(\sigma)+1], \tag{6}$$

which is the cost of the exact preparation of σ under quantum operations preserving the positivity of the partial transpose (PPT). $E_{\mathrm{PPT}}(\sigma)$, similarly to $E_{\mathrm{form}}(\sigma)$ and $E(\sigma)$, gives an upper bound of the entanglement of distillation [27]. As shown by Vidal and Werner [28], the negativity is a convex function; however, $E_{\mathrm{PPT}}(\sigma)$ is *not* convex as a combination of the convex $N(\sigma)$ and the concave logarithmic function. Nevertheless, $E_{\mathrm{PPT}}(\sigma)$ satisfies additivity. For a pure state $|\psi_{\mathrm{P}}\rangle$, it holds that

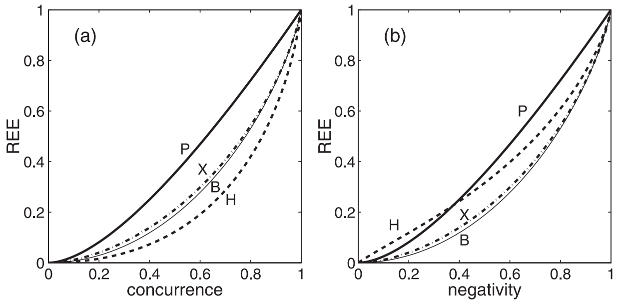

Figure 2. REE versus (a) concurrence and (b) negativity for the boundary states in figure 1(c).

$C(|\psi_{\mathrm{P}}\rangle) = N(|\psi_{\mathrm{P}}\rangle)$ but $E_{\mathrm{PPT}}(|\psi_{\mathrm{P}}\rangle) \geqslant E_{\mathrm{form}}(|\psi_{\mathrm{P}}\rangle)$, where equality holds for separable and maximally entangled states. For these reasons, we will apply concurrence and negativity instead of E_{form} and E_{PPT}.

2. Numerical comparison of state orderings

In previous works much attention was devoted to the ordering problem for the concurrence versus the negativity [2, 3, 9–11]. Here, we will study analytically the ordering of two-qubit states imposed by the REE in comparison to the other two measures. But first let us show the violation of condition (1) by numerical simulation. We have generated 'randomly' 10^5 two-qubit states according to the method described by Życzkowski *et al* [29, 30] and applied, e.g., by Eisert and Plenio [2]. The results are shown in figure 1, where for each state σ generated we have plotted $E(\sigma)$ versus $C(\sigma)$, $E(\sigma)$ versus $N(\sigma)$ and $N(\sigma)$ versus $C(\sigma)$. It is worth noting that the apparent saw-like irregularity of the distribution of states (along the x-axes) is an artefact resulting from the modification of the original Życzkowski *et al* method. That is, we have performed simulations sequentially in ten rounds and during the kth round we plotted the three entanglement measures only for those σ for which $C(\sigma)$ was greater than $(k-1)/10$. The speed-up of this biased simulation is a result of fast procedures for calculating the negativity or concurrence and very inefficient ones for calculating the REE [13, 16, 17, 26]. Our sequential method could be applied since the main goal for generating states was to check efficiently the boundaries of the depicted regions but not the distribution of states.

The bounded regions containing all the states generated, as shown in figure 1 and for clarity redrawn in figure 2, reveal the ordering problem as a result of 'the lack of precision with which one entanglement measure characterizes the other' [7]. By simply generalizing the interpretation given by us in [11] to include any two ($E^{(1)}$ and $E^{(2)}$) of the entanglement measures studied, one can conclude that for any partially entangled state σ' there are infinitely many partially entangled states σ for which the Eisert–Plenio condition, given by (1), is violated. To demonstrate this result explicitly for a given state σ', it is useful to plot $[E^{(2)}(\sigma) - E^{(2)}(\sigma')]$ versus $[E^{(1)}(\sigma) - E^{(1)}(\sigma')]$ as shown in figure 3. Then the state σ corresponding to any point in the regions II and IV is inconsistently ordered with σ' with respect to the measures $E^{(1)}$ and $E^{(2)}$. In contrast, the states σ, corresponding to any point in the regions I and III, and σ' are consistently ordered by $E^{(1)}$ and $E^{(2)}$.

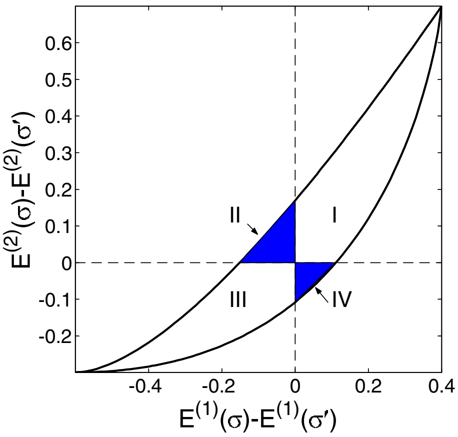

Figure 3. How to find states either satisfying or violating condition (1): all states σ for a given state σ' for which the chosen measures $E^{(1)}$ and $E^{(2)}$ impose the same (opposite) order correspond to points in regions I and III (II and IV).

The probability $P_{\rm ent}$ that a randomly generated two-qubit mixed state is entangled can be estimated as $P_{\rm ent} \approx 0.368 \pm 0.002$ [30] or $P_{\rm ent} \approx 0.365 \pm 0.001$ [2]. However, the probability $P_{\rm viol}$ that a randomly generated pair of two-qubit states violates condition (1) for concurrence and negativity is much less than $P_{\rm ent}$ and estimated as $P_{\rm viol} \approx 0.047 \pm 0.001$ [2]. Since the numerical analysis of Eisert and Plenio [2] and by the power of the Virmani–Plenio theorem [4] we know about the existence of states violating condition (1). But it is not a trivial task to find analytical examples of such states, especially in the case of the orderings imposed by the REE in comparison to other entanglement measures. We believe that it is not only a mathematical problem of classification of states with respect to various entanglement measures but it can shed more light on subtle physical aspects of the entanglement measures including their operational interpretation. By means of a comparison given in the following sections, we will find states exhibiting very surprising properties. In particular, we will show that states σ' and σ'' can have the same negativity, $N(\sigma') = N(\sigma'')$, the same concurrence, $C(\sigma') = C(\sigma'')$, but still different REEs, $E(\sigma') \neq E(\sigma'')$. A deeper analysis of such states can be useful in studies of properties of a given entanglement measure (in this example, the REE) under operations preserving other entanglement measures (here, the entanglement of formation and the PPT entanglement cost). Thus, we believe that it is meaningful to make an analytical study of the violation of condition (1), as will be presented in greater detail in the following sections.

3. Boundary states

The extreme violation of (1) occurs if one of the states corresponds to a point at the upper bound and the other at the lower bound. Thus, for a comparison of different orderings, it is essential to describe the states at the boundaries.

The upper bounds in figure 1 marked by P correspond to two-qubit pure states

$$|\psi_{\rm P}\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle, \qquad (7)$$

where a, b, c, d are the normalized complex amplitudes. The concurrence and negativity are equal to each other and given by

$$C(|\psi_{\rm P}\rangle) = N(|\psi_{\rm P}\rangle) = 2|ad - bc|. \qquad (8)$$

As shown by Verstraete *et al* [5], the negativity of any state σ can never exceed its concurrence (see figure 1(c)), and this bound is reached for the set of states for which the eigenvector of the partial transpose of σ, corresponding to the negative eigenvalue, is a Bell state. Evidently, pure states belong to the Verstraete *et al* set of states. For a pure state the REE is equal to the entanglement of formation, and thus is simply given by Wootters' relation (4) since $E(|\psi_{\rm P}\rangle) = E_{\rm form}(|\psi_{\rm P}\rangle)$. In general, it holds that $E_{\rm form}(\sigma) \geqslant E(\sigma)$ [13], and the REE for pure states gives the upper bound of the REE versus concurrence [5]. We have also conjectured in [31], on the basis of numerical simulations similar to those presented in figure 1(b), that the upper bound of the REE versus negativity N is reached by pure states for $N \geqslant N_0 \equiv 0.3770\cdots$.

Surprisingly, the REE versus N for pure states can be exceeded by other states if $N < N_0$, as was shown in [31] by the so-called Horodecki states, which are mixtures of the maximally entangled state, say the singlet state $|\psi_-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ and a separable state orthogonal to it, say $|00\rangle$, i.e. [1]

$$\sigma_{\rm H} = C|\psi_-\rangle\langle\psi_-| + (1 - C)|00\rangle\langle 00| \qquad (9)$$

for which the concurrence and negativity are given, respectively, by

$$C(\sigma_{\rm H}) = C, \qquad (10a)$$

$$N(\sigma_{\rm H}) = \sqrt{(1 - C)^2 + C^2} - (1 - C). \qquad (10b)$$

Verstraete *et al* [5] proved that a function of the form (10*b*) determines the lower bound of the negativity versus concurrence for any state σ (see curve H in figure 1(c)). On the other hand, the REE versus concurrence for Horodecki states is given by [13]

$$E(\sigma_{\rm H}) = (C - 2)\,{\rm lg}(1 - C/2) + (1 - C)\,{\rm lg}(1 - C). \qquad (11)$$

By replacing C by $\sqrt{2N(1 + N)} - N$ in (11), one gets an explicit dependence of $E(\sigma_{\rm H})$ on the negativity $N(\sigma_{\rm H})$ [31]. It was conjectured that the REE for the Horodecki states describes the lower bound of the REE versus concurrence [13], as shown by curve H in figures 1(a) and 2(a), and also conjectured [31] that it gives the upper bound of the REE versus negativity if $N \leqslant N_0$ as seen in figures 1(b) and 2(b) [31]. The ordering violation for any two of the three entanglement measures can be shown for a pair of the Horodecki and pure states, say σ' and σ, if one of the states is partially entangled ($0 < E^{(1)}(\sigma') < 1$) and σ is properly chosen according to the rule shown in figure 3 with an exception for the following case: if one of the states in the pair of the Horodecki and pure states has the negativity equal to N_0, then the ordering imposed by the REE and negativity for these states is always consistent as required by condition (1).

The lower bound in figure 1(b) and the upper bound in figure 1(c) correspond to the Bell diagonal state (labelled by B), given by

$$\sigma_{\rm B} = \sum_{i=1}^{4} \lambda_i |\beta_i\rangle\langle\beta_i| \qquad (12)$$

with the largest eigenvalue $\max_j \lambda_j \equiv (1+C)/2 \geqslant 1/2$, where $\sum_j \lambda_j = 1$ and $|\beta_i\rangle$ are the Bell states. The negativity and concurrence are the same and given by

$$C(\sigma_B) = N(\sigma_B) = C; \tag{13}$$

thus σ_B, similarly to the pure states case, belongs to the Verstraete *et al* set of states maximizing the negativity for a given concurrence. For the Bell diagonal states, the REE versus the concurrence (and the negativity) reads as [12]

$$E(\sigma_B) = 1 - h((1+C)/2) \\ = \tfrac{1}{2}\left[(1+C)\lg(1+C) + (1-C)\lg(1-C)\right]. \tag{14}$$

If $\max_j \lambda_j \leqslant 1/2$ then the state is separable; thus $C(\sigma_B) = N(\sigma_B) = E(\sigma_B) = 0$. As an example of (12), one can analyse the Werner state [32]

$$\sigma_W = \frac{1+2C}{3}|\psi_-\rangle\langle\psi_-| + \frac{1-C}{6} I \otimes I, \tag{15}$$

where $0 \leqslant C \leqslant 1$; I is the identity operator of a single qubit. Our choice of parametrization of (1) leads to straightforward expressions for the negativity and concurrence given by (13). The results of our simulation of 10^5 random states presented in figure 1(b) confirm our conjecture in [31] that the lower bound of the REE versus negativity is determined by the Bell diagonal states. Nevertheless, to our knowledge, this conjecture and the other proposed by Verstraete *et al* [5] on the lower bound of the REE versus concurrence have not been proved yet [22]. By contrast, it is easy to prove, by applying local random rotations to both qubits [21], that the lower bound of the REE versus fidelity is reached by the Bell diagonal states [13]. It is worth noting that the REE versus concurrence for σ_B is not extreme, as shown by curve B in figure 2(a).

Let us analyse another state corresponding to the upper bound for N versus C, but neither reaching the bounds for E versus C nor those for E versus N. The state is defined as a MES, say the singlet state, mixed with $|01\rangle$ as follows:

$$\sigma_X = C|\psi_-\rangle\langle\psi_-| + (1-C)|01\rangle\langle 01| \tag{16}$$

for which one gets

$$C(\sigma_X) = N(\sigma_X) = C. \tag{17}$$

The eigenvalues of the partially transposed σ_X are $\{1 - C/2, -C/2, C/2, C/2\}$ and they correspond to the eigenvectors given by $\{|01\rangle, |\phi_+\rangle, |\phi_-\rangle, |10\rangle\}$, where $|\phi_\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$. Thus, the Verstraete condition for states with equal concurrence and negativity is fulfilled for the state σ_X, as the negative eigenvalue $-C/2$ corresponds to the Bell state. The separable state $\bar{\rho}_X$ closest to σ_X was found by Vedral and Plenio [13] as $\bar{\rho}_X = (1 - C/2)|01\rangle\langle 01| + C/2|10\rangle\langle 10|$, which enables calculation of the following REE:

$$E(\sigma_X) = h(C/2) - h(r/2), \tag{18}$$

where $r = 1 + \sqrt{(1-C)^2 + C^2}$. Although (17) describes the upper bound for N versus C, (18) differs from the extreme expressions for E versus C and E versus N given for the pure, Horodecki and Bell diagonal states. Figures 2(a) and (b) show clearly the differences.

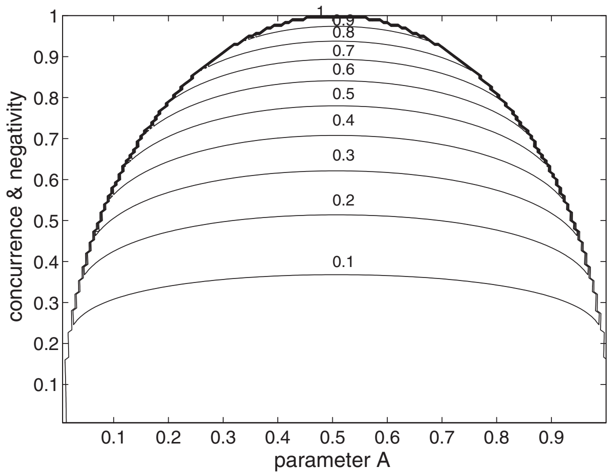

Figure 4. A contour plot of REE $E(\sigma_Y)$ as a function of $C(\sigma_Y) = N(\sigma_Y) = C$ and parameter A according to (21).

We will also analyse the states dependent on two parameters

$$\sigma_Y = A|01\rangle\langle 01| + (1-A)|10\rangle\langle 10| + \frac{C}{2}(|01\rangle\langle 10| + |10\rangle\langle 01|) \tag{19}$$

assuming that $C \leqslant 2\sqrt{A(1-A)}$ to ensure that σ_Y is positive semi-definite. States of the form given by (19) can be obtained by mixing a pure state $|\psi_P\rangle$ with the separable state $\bar{\rho}_P$ closest to $|\psi_P\rangle$ [31]. This mixing leaves the closest separable state unchanged, as implied by the Vedral–Plenio theorem [13]. The eigenvalues of the partial transpose of σ_Y are $\{1-A, A, -C/2, C/2\}$, which correspond to the following eigenvectors: $\{|10\rangle, |01\rangle, |\phi_-\rangle, |\phi_+\rangle\}$, respectively. Thus, the negative eigenvalue $-C/2$ corresponds to the Bell state $|\phi_-\rangle$, which implies that σ_Y belongs to the Verstraete *et al* set of states with equal negativity and concurrence,

$$C(\sigma_Y) = N(\sigma_Y) = C. \tag{20}$$

The REE for state (19) reads as

$$E(\sigma_Y) = h(A) - h\left(\tfrac{1}{2}\left[1 + \sqrt{(1-2A)^2 + C^2}\right]\right) \tag{21}$$

which was obtained with the help of the closest separable state $\bar{\rho}_P = A|01\rangle\langle 01| + (1-A)|10\rangle\langle 10|$ given in [13]. The contour plot of $E(\sigma_Y)$ is shown in figure 4. The states (19), independent of parameter A, are the upper bound states for N versus C. By changing A, the states (19) transform from the pure states into Bell diagonal states; thus they can become the upper bound states both for E versus C and E versus $N \geqslant N_0$, as well as the lower bound states for E versus N. In general, a state corresponding to any point between curves P and B in figures 2(a) and (b) can be given by (19).

4. Analytical comparison of state orderings

By analysing pairs of states discussed in the previous section and by applying the rule shown in figure 3 we can easily find analytical explicit examples of states violating condition (1) for any two measures out of the triple, when the third measure

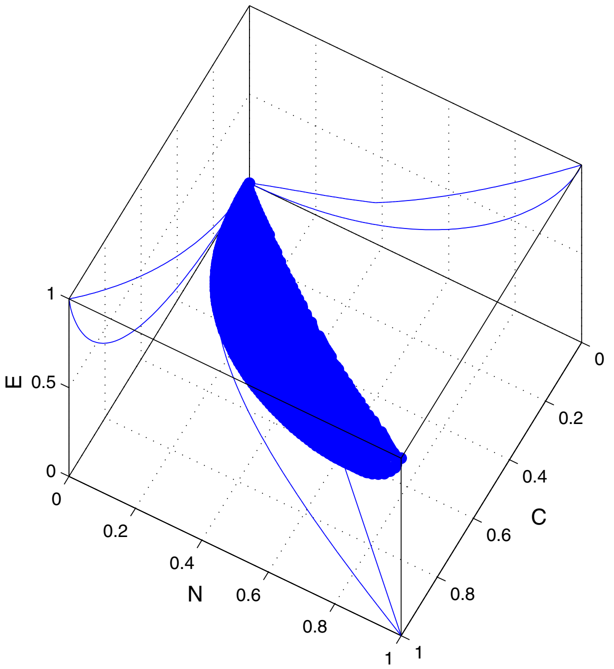

Figure 5. States σ characterized by $[C(\sigma), N(\sigma), E(\sigma)]$ lie in the solid crescent-like region with its projections into the planes shown in figure 1. All classes of state pairs from table 1 can be found by analysing pairs of points at various cross sections of the region.

Table 1. All possible different predictions of the state orderings imposed by the REE, concurrence and negativity. As explained in the text, the remaining 13 classes of state pairs can be obtained from the listed classes just by interchanging definitions of σ' and σ''. Asterisks denote the classes for which we were not able to find examples.

Class	Concurrences	Negativities	REEs
1	$C(\sigma') < C(\sigma'')$,	$N(\sigma') < N(\sigma'')$,	$E(\sigma') < E(\sigma'')$
2	$C(\sigma') < C(\sigma'')$,	$N(\sigma') > N(\sigma'')$,	$E(\sigma') < E(\sigma'')$
3	$C(\sigma') > C(\sigma'')$,	$N(\sigma') < N(\sigma'')$,	$E(\sigma') < E(\sigma'')$
4	$C(\sigma') < C(\sigma'')$,	$N(\sigma') < N(\sigma'')$,	$E(\sigma') > E(\sigma'')$
5	$C(\sigma') = C(\sigma'')$,	$N(\sigma') = N(\sigma'')$,	$E(\sigma') = E(\sigma'')$
6	$C(\sigma') < C(\sigma'')$,	$N(\sigma') = N(\sigma'')$,	$E(\sigma') < E(\sigma'')$
7	$C(\sigma') = C(\sigma'')$,	$N(\sigma') < N(\sigma'')$,	$E(\sigma') < E(\sigma'')$
8	$C(\sigma') < C(\sigma'')$,	$N(\sigma') < N(\sigma'')$,	$E(\sigma') = E(\sigma'')$
9	$C(\sigma') = C(\sigma'')$,	$N(\sigma') = N(\sigma'')$,	$E(\sigma') < E(\sigma'')$
10	$C(\sigma') < C(\sigma'')$,	$N(\sigma') = N(\sigma'')$,	$E(\sigma') = E(\sigma'')$
*11	$C(\sigma') = C(\sigma'')$,	$N(\sigma') < N(\sigma'')$,	$E(\sigma') = E(\sigma'')$
*12	$C(\sigma') > C(\sigma'')$,	$N(\sigma') = N(\sigma'')$,	$E(\sigma') < E(\sigma'')$
*13	$C(\sigma') = C(\sigma'')$,	$N(\sigma') > N(\sigma'')$,	$E(\sigma') < E(\sigma'')$
14	$C(\sigma') < C(\sigma'')$,	$N(\sigma') > N(\sigma'')$,	$E(\sigma') = E(\sigma'')$

is not analysed. However, the number of classes of state pairs increases to 14, as shown in table 1, on including all possible different predictions of the state orderings imposed by all the three measures simultaneously. The number of classes is given mathematically by permutation with replacement (where the order counts and repetitions are allowed) and equal to 3^3. But we should not count twice the classes defined by opposite inequalities (e.g., class 2 can be equivalently given by $E(\sigma') > E(\sigma'')$, $C(\sigma') > C(\sigma'')$ and $N(\sigma') < N(\sigma'')$) since the definition of states σ' and σ'' can be interchanged. Thus, the number of classes decreases to $(3^3 - 1)/2 + 1 = 14$. One can identify all these classes by analysing pairs of points

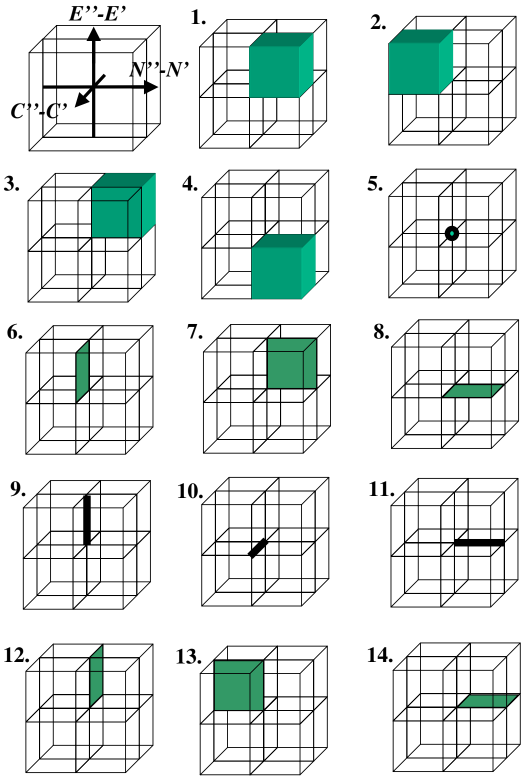

Figure 6. A schematic representation of the 14 classes of state pairs listed in table 1, where $f' = f(\sigma')$ and $f'' = f(\sigma'')$ for $f = C, N, E$. The central point corresponds to a state σ' for which $\Delta = [0, 0, 0]$. A pair of states σ' and σ'', where the latter is represented by any point inside the marked region of the ith ($i = 1, \ldots, 14$) sub-figure, satisfies the inequalities of the ith class in table 1.

in the crescent-like solid region in CNE space shown in figure 5 with the familiar projections into the planes CE (see also figure 1(a)), NE (figure 1(b)) and CN (figure 1(c)). Unfortunately, a graphical illustration of various cross sections of the solid crescent in figure 5 would not be clear enough. Thus, in figure 6, we give a symbolic representation of the 14 classes of table 1 by depicting only small cubes around point $[C(\sigma'), N(\sigma'), E(\sigma')]$ for a given state σ'. In a sense, the cubes are cut inside the solid crescent shown in figure 5.

In the following, we will give explicit examples of the pairs of states satisfying the inequalities listed in table 1. To make the notation compact we write

$$\Delta \equiv [C(\sigma'') - C(\sigma'), N(\sigma'') - N(\sigma'), E(\sigma'') - E(\sigma')].$$

States consistently ordered by all the three measures as required by the Eisert–Plenio condition (1) belong to *class 1*. The vast majority of the randomly generated pairs of two-qubit states belong to this class. The simplest analytical example is a pair of pure states $|\psi_i\rangle = a_i|00\rangle + b_i|01\rangle + c_i|10\rangle + d_i|11\rangle$ $(i = 1, 2)$, for which $|a_1d_1 - b_1c_1| \neq |a_2d_2 - b_2c_2|$. Similarly, by comparing other pairs of states, for example $(\sigma_H(C'), \sigma_H(C''))$, $(\sigma_B(C'), \sigma_B(C''))$ or $(\sigma_X(C'), \sigma_X(C''))$ for $C' \neq C''$, one arrives at the same conclusion. A pair of states from *class 2* can be given, e.g., by the Bell diagonal and Horodecki states for slightly different concurrences (or negativities). For example, if $\sigma_B(C = 0.5)$ and $\sigma_H(C = 0.6)$, then $\Delta =$

$[0.1, -0.179, 0.003]$, or for the same σ_B but σ_H having its negativity equal to 0.4 then $\Delta = [0.158, -0.1, 0.055]$ as required. As an example of the state pair from *class 3*, we choose the Horodecki and pure states such that their negativities are close to N_0. For example, let σ_H have the negativity $N_0 - 0.1$ and $|\psi_P\rangle$ have its coefficients satisfying $2|ad - bc| = N_0$; then $\Delta = [-0.187, 0.1, 0.064]$. On choosing a pure state with concurrence $C' = 0.625\cdots$ and the Horodecki state for $C'' = 0.846\cdots \equiv C_0$, we observe that their REEs are the same. Then, an example of the state pair from *class 4* can be given by the above pure state and the Horodecki state with its concurrence slightly less than C_0, say $C(\sigma_H) = C_0 - 0.02$, which implies that $\Delta = [0.200, 0.044, -0.037]$ as required. The classes 1–4 are defined solely by sharp inequalities, and thus they are crucial in our comparison of different state orderings.

Now, we will present a more subtle comparison to include the classes when some of the entanglement measures are equal to each other for different states. *Class 5* is interesting enough to be analysed separately in the next section. An example of the state pair from *class 6* can be given by the Bell diagonal and Horodecki states with the same negativities, say equal to 1/2, which implies that $\Delta = [0.225, 0, 0.127]$. Also a member of *class 7* can be given by the above states but for the same concurrences, say $C = 0.5$, which implies that $-\Delta = [0, 0.293, 0.066]$. Simple examples of the state pairs from *classes 6* and *7* can also be found by considering the following state:

$$\sigma_Z(C,N) = \tfrac{1}{2}[(1-\alpha)(|01\rangle\langle 01| + |10\rangle\langle 10|) + C(|01\rangle\langle 10| + |10\rangle\langle 01|) + 2\alpha|00\rangle\langle 00|] \quad (22)$$

for $N > 0$ and $C \in \langle N, \sqrt{2N(N+1)} - N\rangle$, where $\alpha = (C^2 - N^2)/(2N)$. The range-limited C ensures semi-definiteness of σ_Z. State (22) can be generated by mixing the Horodecki state σ_H with the separable state $\bar{\rho}_H$ closest to σ_H given by Vedral and Plenio [13] (for details see [31]). We note that the coefficients C and N in (22) are chosen such that

$$C(\sigma_Z) = C, \qquad N(\sigma_Z) = N. \quad (23)$$

Then, we can write the REE as follows:

$$E(\sigma_Z) = h_3\left((1+\alpha)\beta, \tfrac{1}{2}(1+\alpha)(1-2\beta) + \beta C\right) - h_3(\alpha, \tfrac{1}{2}(1-\alpha+C)), \quad (24)$$

where $\beta = \alpha(1+\alpha)/[(1+\alpha)^2 - C^2]$ and $h_3(x_1, x_2) = -\sum_{i=1}^{3} x_i \lg x_i$ with $x_3 = 1 - x_1 - x_2$. By changing C and N separately, we can obtain σ_Z with a desired REE. For example, by fixing the negativity, we get the state pair corresponding to *class 6*, as shown by the contours of constant negativity in figure 7(a). On the other hand, by fixing the concurrence, the resulting states σ_Z satisfy the conditions for *class 7*, as presented by the contours of constant concurrence in figure 7(b).

To *class 8* belongs a pair of, e.g., the pure state with concurrence $0.625\cdots$ and the Horodecki state with $C = 0.846\cdots$; then it holds that $E(|\psi_P\rangle) = E(\sigma_H) = 0.5$, and $\Delta = [0.220, 0.080, 0]$ as required. To find an exemplary member of *class 9*, one can compare a pure state and any other state from the Verstraete *et al* set of states (including σ_B, σ_X or σ_Y) with the same concurrence, which means also the

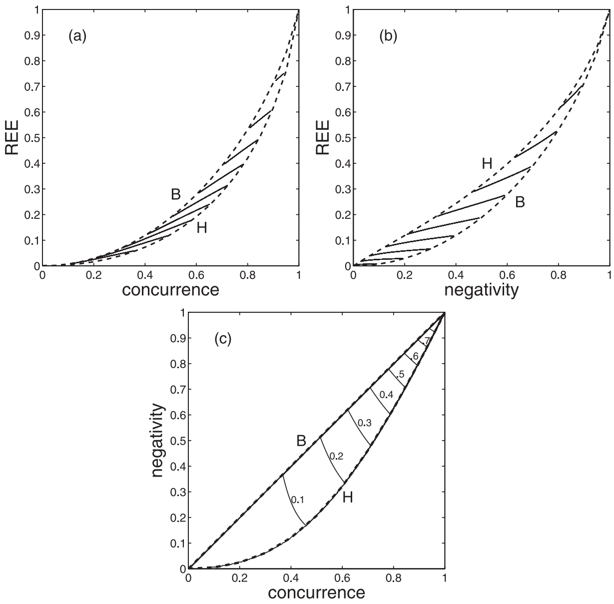

Figure 7. Contour plots of the entanglement measures for σ_Z: (a) negativity $N(\sigma_Z)$ as a function of $C(\sigma_Z)$ and $E(\sigma_Z)$, (b) concurrence $C(\sigma_Z)$ as a function of $N(\sigma_Z)$ and $E(\sigma_Z)$ and (c) REE $E(\sigma_Z)$ as a function of $C(\sigma_Z)$ and $N(\sigma_Z)$. The contours are depicted at values of $0.1, 0.2, \ldots, 1$ from the bottom left corner to the upper right corner.

same negativity. For example, for $C(|\psi_P\rangle) = C(\sigma_B) = 1/2$ one gets $\Delta = [0, 0, 0.189]$. As regards *class 10*, we can compare the pure and Horodecki states with the same negativity $N = N_0$, which implies that $E(|\psi_P\rangle) = E(\sigma_H)$. Thus, we have $\Delta = [0.265, 0, 0]$. Unfortunately, by comparing the states discussed in this section, we have not found examples of state pairs from *classes 11–13*. But we can give a few exemplary members of *class 14*. For example, by comparing the Bell diagonal state for $C' = 0.779\cdots$ and the Horodecki state for $C'' = 0.846\cdots$ we find that $E(\sigma_B) = E(\sigma_H) = 0.5$, while their negativities and concurrences violate condition (1) to the following degrees: $\Delta = [0.066, -0.074, 0]$. Also by analysing figure 7(c) for any two points at the same contour of constant REE, we find exemplary state pairs from *class 14*. Thus, we have presented simple analytical examples of the states satisfying 11 out of 14 classes listed in table 1.

5. States with the same E, C and N

Here, we will analyse examples of inequivalent states $\sigma' \neq \sigma''$, which have the same degree of entanglement according to E, C and N, thus corresponding to *class 5* in table 1. It is tempting to choose simply two different pure states with their coefficients satisfying $|a_1 d_1 - b_1 c_1| = |a_2 d_2 - b_2 c_2|$, which guarantees the fulfilment of the equalities required for this class. However, such pure states can be transformed into each other by local operations. To show this, first we note that any pure state, given by (7), can be transformed by local rotations into the superposition $|\tilde{\psi}_P(p)\rangle = \sqrt{p}|01\rangle + \sqrt{1-p}|10\rangle$ $(0 \leqslant p \leqslant 1)$, for which the concurrence and negativity are equal to $2\sqrt{p(1-p)}$, as a special case of (8). The same value of these entanglement measures occurs also for $|\tilde{\psi}_P(1-p)\rangle$, but this state can be transformed into $|\tilde{\psi}_P(p)\rangle$ by applying a NOT gate to each of the qubits. Thus, we have shown that pure states are not good examples of the state pairs from *class 5*. Then, let us choose, e.g., two different Bell diagonal states but with the same largest eigenvalue greater than 1/2. By virtue of (13) and (14), we conclude that these states have the same degree of entanglement according to the REE, concurrence and negativity. However, as we will show in the following, they can violate the Bell inequality to different degrees.

The maximum possible violation of the Bell inequality in the Clauser–Horne–Shimony–Holt (CHSH) form [33]

$$|\langle\mathcal{B}\rangle_\sigma| = |\mathcal{E}(\phi_1,\phi_2)+\mathcal{E}(\phi_1',\phi_2)+\mathcal{E}(\phi_1,\phi_2')-\mathcal{E}(\phi_1',\phi_2')| \leqslant 2 \tag{25}$$

for a two-qubit state σ is given by [34]

$$\max_{\mathcal{B}}\langle\mathcal{B}\rangle_\sigma = 2\sqrt{M(\sigma)}. \tag{26}$$

Here, $\mathcal{B}$ is the Bell operator, ϕ_i, ϕ_i' are two dichotomic variables of the ith qubit and $\mathcal{E}(\phi_1,\phi_2)$ is the expectation value of the joint measurement of ϕ_1 and ϕ_2, and so on for the other expectation values. The quantity $M(\sigma)$ is the sum of the two largest eigenvalues of $T_pT_p^\dagger$, where T_p is the 3×3 matrix formed by the elements $t_{nm} = \mathrm{Tr}(\sigma\sigma^{(n)}\otimes\sigma^{(m)})$ given in terms of the Pauli matrices $\sigma^{(j)}$. Inequality (25) is satisfied if and only if $M(\sigma)\leqslant 1$ [34]. As shown in [10], for any pure state $|\psi_{\mathrm{P}}\rangle$, the Bell inequality violation parameter $M(\sigma)$ is closely related to the concurrence and negativity as follows:

$$\sqrt{\max\{0, M(|\psi_{\mathrm{P}}\rangle)-1\}} = C(|\psi_{\mathrm{P}}\rangle) = N(|\psi_{\mathrm{P}}\rangle). \tag{27}$$

We find that $M(\sigma)$ for the Bell diagonal state reads as

$$M(\sigma_{\mathrm{B}}) = 2\max_{(i,j,k)}[(\lambda_i-\lambda_j)^2+(\lambda_k-\lambda_4)^2], \tag{28}$$

where subscripts (i, j, k) change over cyclic permutations of (1, 2, 3). Concluding, the Bell inequality violation depends on all λ_is, while the entanglement measures E, C and N depend solely on the largest λ_i. Thus, as an example of a state pair from *class 5*, we can choose two Bell diagonal states σ_{B}' and σ_{B}'' with only the largest eigenvalues being the same and greater than 1/2 for both states, which implies that the states cannot be transformed into each other by LOCC operations but still have the same degrees of entanglement: $E(\sigma_{\mathrm{B}}') = E(\sigma_{\mathrm{B}}'')$, $C(\sigma_{\mathrm{B}}') = C(\sigma_{\mathrm{B}}'')$ and $N(\sigma_{\mathrm{B}}') = N(\sigma_{\mathrm{B}}'')$.

6. Conclusions

We have analysed the problem of inconsistency in ordering states with the entanglement measures. The problem was raised by Eisert and Plenio [2] on the basis of the numerical example of the concurrence and negativity and then studied by others [3, 5–11]. The ordering problem is closely related to the existence of the upper and lower bounds of one entanglement measure versus the other [5, 7, 11, 31]. Here, we presented analytical examples of the pairs of states consistently and inconsistently ordered by the relative entropy of entanglement in comparison to the concurrence and negativity. In particular, we have found examples of states for which any of the measures imposes order opposite to that given by the other two measures, which corresponds to *classes 1–4* in table 1. We have also identified: pairs of states with, in particular, (i) the same concurrences and negativities but different REEs (corresponding to *class 9*), (ii) the same REEs and negativities but different concurrences (*class 10*), (iii) the same REEs but different and oppositely ordered concurrences and negativities (*class 14*); and (iv) states having the same three entanglement measures (*class 5*), but still violating the Bell–CHSH inequality to different degrees.

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