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Decoherence of two maximally entangled qubits in a lossy nonlinear cavity

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Abstract

Decoherence effect on quantum entanglement of two optical qubits in a lossy cavity interacting with a nonlinear medium (Kerr nonlinearity) is analysed. The qubits are assumed to be initially in the maximally entangled states (Bell or Bell-like states) or the maximally entangled mixed states, on the example of Werner and Werner-like states. Two kinds of measures of the entanglement are considered: the concurrence to describe a decay of the entanglement of formation of the qubits, and the negativity to determine a decay of the entanglement cost under positive-partial-transpose-preserving operations. It is observed that the Kerr nonlinearity, in the discussed decoherence model, does not affect the entanglement of the qubits initially in the Bell or Werner states, although the evolution of the qubits can depend on this nonlinearity explicitly. However, it is shown that for the initial Bell-like state and the corresponding Werner-like state, the loss of the entanglement can be periodically reduced by inserting the Kerr nonlinearity in the lossy cavity. Moreover, the relativity of the entanglement measures is demonstrated, to our knowledge for the first time, as a result of a physical process.

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1. Introduction

Decoherence, resulting from the unavoidable and irreversible coupling of a quantum system to its environment, turns a correlated quantum state of the system into a classical statistical mixture [1]. Decoherence, causing usually a loss of quantum entanglement, is one of the major limitations of practical capabilities of quantum computers [2]. Thus, the analysis of the dynamics of entangled quantum two-level systems (qubits) coupled to the environment, represented by a thermal reservoir, is of particular importance. In this paper, we will study the loss of the entanglement due to a dissipative nonlinear interaction of two optical qubits, which are implemented by superpositions of vacuum and single-photon states of two-cavity modes, and assumed to be initially in the maximally entangled states (MESs) or the maximally entangled mixed states (MEMSs).

It is a well-accepted fact that there is no unique way to quantify mixed-state entanglement and thus various measures with different operational interpretations have been proposed to describe different aspects of the entanglement. We will apply the concurrence [3]—a measure related to the entanglement of formation [4], and the negativity [5–7]—a measure corresponding to an operation-limited entanglement cost [8, 9]. We have chosen these particular measures as they are associated with a physical point of view and, moreover, can easily be calculated in contrast to other measures including the entanglement of distillation or the relative entropy of entanglement.

Our analysis is related to a new regime of quantum nonlinear optics involving highly efficient nonlinear interactions between very weak optical fields, which has been recently demonstrated experimentally in, e.g., dense atomic media by using an electromagnetically-induced transparency (EIT) to resonantly enhance nonlinearities (for a review see [10]). In particular, the observation of giant Kerr nonlinearities has been predicted [11] and first measured in an ultracold gas of sodium atoms to be $\sim 10^6$ greater than those in conventional optical materials [12]. Physical realizations of a Kerr nonlinear cavity enabling strong interaction of photons were suggested by Imamoğlu *et al* [11, 13] and then studied by others [14–18]. Motivated by these advances, there is increasing interest to apply the Kerr nonlinearities for quantum information purposes [19], including the problem of the generation of highly entangled states (see [15, 20] and references therein). Nevertheless, the effects of decoherence on the entanglement of fields interacting via the Kerr nonlinearity have not been discussed in more detail yet.

This paper is organized as follows. In section 2, we define the entanglement measures to be used in our description of decoherence. The model and its solution for two optical qubits in a lossy nonlinear cavity are presented in section 3. The main results concerning the decoherence of the qubits being initially in the maximally entangled states and the maximally entangled mixed states are presented in sections 4 and 5, respectively. A physical implementation of the model and discussion of the results are given in section 6.

2. Entanglement measures

We will apply two measures of entanglement to analyse the effect of decoherence on the entangled qubit states. The first measure is the concurrence defined for two qubits as [3]

$$C(\hat{\rho}) = \max\left\{2\max_i \lambda_i - \sum_{i=1}^{4}\lambda_i, 0\right\} \tag{1}$$

where λ_i are the square roots of the eigenvalues of the matrix $\hat{\rho}(\hat{\sigma}_{1y} \otimes \hat{\sigma}_{2y})\hat{\rho}^*(\hat{\sigma}_{1y} \otimes \hat{\sigma}_{2y})$, where $\hat{\sigma}_{jy}$ is a Pauli spin matrix of the jth qubit and the asterisk denotes complex conjugation. The entanglement of formation, $E_F(\hat{\rho})$, which characterizes the amount of entanglement necessary to create the entangled state [4], is for two qubits given by a simple monotonic function of the concurrence [3]

$$E_F(\hat{\rho}) = H\left\{\tfrac{1}{2}\left[1 + \sqrt{1 - C(\hat{\rho})^2}\right]\right\} \tag{2}$$

where $H\{x\} = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy. The entanglement of formation and, equivalently, the concurrence vanish for an unentangled state, and are equal to 1 for a maximally entangled state. A measure associated with the entanglement of formation is the entanglement cost defined as [4] $\lim_{n\to\infty} E_F(\hat{\rho}^{\otimes n})/n$ which, in general, is quite difficult to calculate. Thus, for simplicity, we will describe the entanglement cost limited to a special class of operations to be specified in the following.

Another useful measure of the entanglement is the negativity [6, 7], which corresponds to a quantitative version of the Peres–Horodecki criterion [5]. We adopt here the following definition:

$$\mathcal{N}(\hat{\rho}) = 2\max\left(0, -\sum_j \mu_j\right) \tag{3}$$

where the sum is taken over the negative eigenvalues μ_j of the partial transpose $\hat{\rho}^{\mathrm{T}_A}$ of the density matrix $\hat{\rho}$ of the system. For two-qubit pure or mixed states, the sum in (3) can be skipped as $\hat{\rho}^{\mathrm{T}_A}$ has at most one negative eigenvalue [21]. The negativity satisfies the standard conditions for a useful measure of the entanglement [22, 23]. For two-qubit states, the negativity, defined by (3), becomes 1 for a MES and vanishes for an unentangled state, the same as the concurrence. Recently, Audenaert *et al* [8] and, supplementarily, Ishizaka [9] have provided an operational interpretation of the logarithmic negativity, defined by [23]

$$E_N(\hat{\rho}) = \log_2[N(\hat{\rho}) + 1], \tag{4}$$

as a measure of the entanglement cost for the exact preparation of a two-qubit quantum state $\hat{\rho}$ under quantum operations preserving the positivity of the partial transpose (PPT).

For an arbitrary two-qubit pure state

$$|\Psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle \tag{5}$$

with the normalized complex amplitudes c_{ij}, we have the concurrence and negativity equal to each other and given by a simple formula

$$\mathcal{N}_\Psi = C_\Psi = 2|c_{00}c_{11} - c_{01}c_{10}|. \tag{6}$$

However, for qubits in a mixed state, the entanglement measures are usually different. In general, the inequality $\mathcal{N}(\hat{\rho}) \leqslant C(\hat{\rho})$ holds for an arbitrary two-qubit state $\hat{\rho}$ as first observed by numerical investigation by Eisert and Plenio [7] and Życzkowski [24], and then proved by Verstraete *et al* [25].

Eisert and Plenio [7] raised an intriguing problem of the relativity of entanglement measures: if according to one measure of the entanglement the state $\hat{\rho}_1$ is more entangled than $\hat{\rho}_2$ then does it imply that $\hat{\rho}_1$ is also more entangled than $\hat{\rho}_2$ according to another entanglement measure? By the Monte Carlo simulation of thousands of two-qubit states, it was observed that indeed the condition

$$C(\hat{\rho}_1) < C(\hat{\rho}_2) \Leftrightarrow \mathcal{N}(\hat{\rho}_1) < \mathcal{N}(\hat{\rho}_2) \tag{7}$$

can be violated by some states $\hat{\rho}_1$ and $\hat{\rho}_2$ [7, 24, 26]. It should be stressed that this odd looking property is physically sound since such incomparable states $\hat{\rho}_1$ and $\hat{\rho}_2$ cannot be transformed into each other with unit efficiency by any local quantum operations and classical communication (LQCC). In general, all good asymptotic entanglement measures are either identical or put different orderings of quantum states as implied by the requirements of equivalence and continuity of the measures on pure states [27]. Thus, by comparing various entanglement measures defined to examine different methods of entanglement preparation and/or its use, one indeed can arrive at the problem of different state orderings imposed by the measures. The only alternative to avoid the state-ordering ambiguity is to declare one entanglement measure for mixed states as the unique one, but this would preclude us from examining the problems of how to prepare the entanglement and how to make use of it [28]. Here, we will give simple analytical examples of states differently ordered by the concurrence and negativity, thus we will explicitly demonstrate the relativity of these entanglement measures.

3. Model and its solution

Decoherence effects on optical modes (qubits) in a lossy nonlinear cavity can be described by a model of N-coupled dissipative nonlinear oscillators represented by the following prototype Hamiltonian [29][3]:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\rm NL} + \hat{H}_I \tag{8}$$

where

$$\hat{H}_0 = \hbar \sum_{j=1}^{N} \omega_j \hat{a}_j^\dagger \hat{a}_j + \hbar \sum_k \sum_{j=1}^{N} \Omega_k^{(j)} \big(\hat{b}_k^{(j)}\big)^\dagger \hat{b}_k^{(j)}, \tag{9}$$

$$\hat{H}_{\rm NL} = \hbar \sum_{i,j=1}^{N} \chi_{ij} \hat{a}_i^\dagger \hat{a}_i \hat{a}_j^\dagger \hat{a}_j, \tag{10}$$

$$\hat{H}_I = \hbar \sum_k \sum_{j=1}^{N} \big[g_k^{(j)} \hat{a}_j^\dagger \hat{b}_k^{(j)} + \text{h.c.}\big], \tag{11}$$

and $\hat{a}_j$ is the annihilation operator for the jth system oscillator at the frequency ω_j; $\hat{b}_k^{(j)}$ is the annihilation operator for the kth oscillator in the jth reservoir at the frequency $\Omega_k^{(j)}$; χ_{ij} are the nonlinear self-coupling (for $i = j$) and cross-coupling (for $i \neq j$) constants proportional to a third-order susceptibility of the Kerr nonlinear medium (see section 6) and $g_k^{(j)}$ are the coupling constants of the reservoir oscillators. Dissipation of the system is modelled by its coupling to reservoirs of oscillators as described by the Hamiltonian $\hat{H}_I$. The evolution of the dissipative system under the Markov approximation is governed by the following master equation for the reduced density operator $\hat{\rho}$ in the interaction picture:

$$\frac{\partial}{\partial t}\hat{\rho} = \frac{1}{\mathrm{i}\hbar}[\hat{H}_{\rm NL}, \hat{\rho}] + \sum_{j=1}^{N} \frac{\gamma_j}{2} \big\{\bar{n}_j \big(2\hat{a}_j^\dagger \hat{\rho} \hat{a}_j - \hat{a}_j \hat{a}_j^\dagger \hat{\rho} - \hat{\rho} \hat{a}_j \hat{a}_j^\dagger\big) + (\bar{n}_j + 1)\big(2\hat{a}_j \hat{\rho} \hat{a}_j^\dagger - \hat{a}_j^\dagger \hat{a}_j \hat{\rho} - \hat{\rho} \hat{a}_j^\dagger \hat{a}_j\big)\big\} \tag{12}$$

where $\bar{n}_j$ are the mean thermal occupation numbers and γ_j are the damping constants, which will be assumed the same, $\gamma_1 = \gamma_2 \equiv \gamma$, in the next sections. With the help of a disentangling theorem for $SU(1, 1)$ in thermofield dynamics notation, Chaturvedi and Srinivasan [30] have found a general solution of the master equation (12) both for the quiet ($\bar{n}_j = 0$) and noisy ($\bar{n}_j > 0$) reservoirs. By confining our analysis to the case of only two oscillators ($N = 2$) coupled to the quiet reservoirs, the Chaturvedi–Srinivasan solution for the density matrix elements $\langle m_1, m_2|\hat{\rho}(t)|n_1, n_2\rangle$ in the photon-number basis can be written as

$$\langle m_1, m_2|\hat{\rho}(t)|n_1, n_2\rangle = \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} R_1 R_2 \langle m_1 + p_1, m_2 + p_2|\hat{\rho}(0)|n_1 + p_1, n_2 + p_2\rangle \tag{13}$$

[3] Hamiltonian $\hat{H}_{\rm NL}$ is sometimes defined in the normal-ordered form of $\hat{a}_j^\dagger$ and $\hat{a}_j$. However, such Hamiltonian differs from ours only in terms proportional to $\chi_{jj}\hat{a}_j^\dagger \hat{a}_j$, which can be incorporated in the free Hamiltonian $\hat{H}_0$, so does not effect the entanglement.

where

$$R_j \equiv R_j(m_j, n_j, p_j) = \left[\binom{m_j + p_j}{p_j}\binom{n_j + p_j}{p_j}\right]^{1/2} \left(\frac{\gamma_j}{x_j}[1 - \exp(-x_j t)]\right)^{p_j} \times \exp\left\{\mathrm{i}(\chi_{j1} + \chi_{j2})(m_j - n_j)t - [x_j(m_j + n_j + 1) - \gamma_j]\frac{t}{2}\right\} \tag{14}$$

with $x_j = \gamma_j + 2\mathrm{i}[\chi_{j1}(m_1 - n_1) + \chi_{j2}(m_2 - n_2)]$ and $\binom{q}{p}$ are binomial coefficients. In our scheme, qubits can be represented by the single-cavity modes restricted in the Hilbert space spanned by vacuum and single-photon states (see, e.g., [31]). Then, for the qubit states, solution (13) simplifies to the summations over $p_1, p_2 = 0, 1$ only.

By assuming no dissipation ($\gamma_1 = \gamma_2 = 0$) in our system, the evolution is governed by the unitary operator $\exp(-\mathrm{i}\hat{H}_{\mathrm{NL}}t/\hbar)$. It implies that, for the two qubits initially in a pure state (5), the concurrence and negativity evolve periodically as follows:

$$\mathcal{N}_\Psi(\gamma = 0, t) = C_\Psi(\gamma = 0, t) = 2|\exp(2\mathrm{i}\chi_{12}t)c_{00}(0)c_{11}(0) - c_{01}(0)c_{10}(0)| \tag{15}$$

depending on the cross-coupling χ_{12} but not on the self-coupling constants χ_{11} and χ_{22}. One can observe that the evolution of the qubits in the nonlinear medium can lead to a periodical generation of entangled states. Even for the initial separable state

$$|\Psi(0)\rangle = (d_1|0\rangle_1 + d_2|1\rangle_1) \otimes (d_3|0\rangle_2 + d_4|1\rangle_2), \tag{16}$$

where $|d_1|^2 + |d_2|^2 = |d_3|^2 + |d_4|^2 = 1$ and none of the amplitudes d_i is zero, the concurrence and negativity periodically become positive

$$\mathcal{N}_\Psi(\gamma = 0, t) = C_\Psi(\gamma = 0, t) = 4|d_1 d_2 d_3 d_4 \sin(\chi_{12}t)| \tag{17}$$

which corresponds to the entanglement of up to $H\left\{\frac{1}{2}\left[1 + \sqrt{1 - 16|d_1 d_2 d_3 d_4|^2}\right]\right\}$ ebits. In particular, the initial state (16) with all the amplitudes equal to $1/\sqrt{2}$, i.e.,

$$|\Psi(0)\rangle = \frac{|0\rangle_1 + |1\rangle_1}{\sqrt{2}} \otimes \frac{|0\rangle_2 + |1\rangle_2}{\sqrt{2}} \equiv |+,+\rangle, \tag{18}$$

evolves into a maximally entangled state, defined below by (26), having exactly 1 ebit at the evolution moments $t = (1 + 2n)\pi/(2\chi_{12})$ $(n = 0, 1, \ldots)$. Nevertheless, the MESs are not generated if our system is subjected to dissipation.

4. Decoherence of the maximally entangled states

Let us assume that two qubits are initially in the Bell states

$$|\psi_\pm\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \tag{19}$$

which evolve in the dissipative nonlinear cavity into the following mixed states:

$$\hat{\rho}_{\psi\pm}(t) = \tfrac{1}{2}\{2(1 - g)|00\rangle\langle 00| + g(|01\rangle\langle 01| + |10\rangle\langle 10|) \pm g(\mathrm{e}^{\mathrm{i}(\chi_1 - \chi_2)t}|01\rangle\langle 10| + \text{h.c.})\} \tag{20}$$

where $g = \mathrm{e}^{-\gamma t}$ and $\chi_i \equiv \chi_{ii}$. The evolution is independent of the nonlinear cross-coupling χ_{12} but depends on the self-couplings χ_1 and χ_2. We find that the concurrence is simply given by

$$C_\psi(t) = g \tag{21}$$

and the negativity is

$$\mathcal{N}_\psi(t) = \sqrt{2g^2 - 2g + 1} + g - 1 \tag{22}$$

being independent of the sign in (19). As implied by the form of the density matrices (20), the entanglement measures are independent of any Kerr couplings. On the other hand, the initial Bell states

$$|\phi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \tag{23}$$

evolve in our lossy system into

$$\hat{\rho}_{\phi\pm}(t) = \tfrac{1}{2}\{(2-2g+g^2)|00\rangle\langle 00| + (1-g)g(|01\rangle\langle 01| + |10\rangle\langle 10|) \pm g(\mathrm{e}^{\mathrm{i}(\chi_1+2\chi_{12}+\chi_2)t}|00\rangle\langle 11| + \mathrm{h.c.}) + g^2|11\rangle\langle 11|\}. \tag{24}$$

In contrast to $\hat{\rho}_{\psi\pm}(t)$, the density matrices $\hat{\rho}_{\phi\pm}(t)$ depend on the cross-coupling between the qubits. We find that the concurrence and negativity are the same for any evolution times and any sign in (23) as given by

$$C_{\phi}(t) = \mathcal{N}_{\phi}(t) = g^2 \tag{25}$$

in contrast to $C_{\psi}(t)$ and $\mathcal{N}_{\psi}(t)$, given by (21) and (22), respectively, which are the same at $t = 0$ and $t = \infty$ only. There are the following important properties of the discussed entanglement decays. First, the concurrence (negativity) for the initial Bell states $|\psi_{\pm}\rangle$ decays slower (faster) than that for $|\phi_{\pm}\rangle$, as it holds $C_{\psi}(t) > C_{\phi}(t)$ and $\mathcal{N}_{\psi}(t) < \mathcal{N}_{\phi}(t)$ for any damping constants $\gamma_k > 0$ ($k = 1, 2$) and any moments of time $0 < t < \infty$. Thus, we provide an explicit example of states violating condition (7). Second, in contrast to the density matrices, the entanglement measures are independent of the nonlinear couplings for the initial Bell states (19) and (23). Thus, decoherence-free evolution in the nonlinear cavity does not change the entanglement, i.e., $C_{\psi}(\gamma = 0, t) = \mathcal{N}_{\psi}(\gamma = 0, t) = 1$. Now we will give an example of a maximally entangled two-qubit state evolving in the Kerr medium in such a way that the entanglement depends on the cross-coupling χ_{12}. Let us analyse the following initial state:

$$|\varphi\rangle = \tfrac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) \equiv \frac{1}{\sqrt{2}}(|0,+\rangle + |1,-\rangle) \tag{26}$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. The state $|\varphi\rangle$ is a MES since its concurrence and negativity are equal to 1. For brevity, we neglect the self-couplings, $\chi_1 = \chi_2 = 0$, which do not affect the entanglement. Then the initial pure state $|\varphi\rangle$ evolves in the Kerr medium into the mixed state described by

$$\hat{\rho}_{\varphi}(t) = \frac{1}{4}\begin{bmatrix} (2-g)^2 & h\sqrt{g} & h\sqrt{g} & -fg \\ h^*\sqrt{g} & g(2-g) & g & -fg^{3/2} \\ h^*\sqrt{g} & g & g(2-g) & -fg^{3/2} \\ -f^*g & -f^*g^{3/2} & -f^*g^{3/2} & g^2 \end{bmatrix} \tag{27}$$

given, as usual, in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, and $h = (\gamma fg - 2\mathrm{i}\chi_{12})/(\gamma - 2\mathrm{i}\chi_{12})$, $g = \exp(-\gamma t)$ and $f = \exp(2\mathrm{i}\chi_{12}t)$. Moreover, if we assume no losses in the nonlinear cavity ($\gamma = 0$), then the evolution of the initial state $|\varphi\rangle$ results in the entanglement oscillations described simply by

$$C_{\varphi}(\gamma = 0, t) = \mathcal{N}_{\varphi}(\gamma = 0, t) = |\cos(\chi_{12}t)|, \tag{28}$$

which is in contrast to the case of the initial Bell states $|\phi_{\pm}\rangle$ and $|\psi_{\pm}\rangle$, which evolve without changing their entanglements. The aperiodic decay of the entanglement for the density matrix (27) occurs only if there is no interaction between the qubits and then it is described by

$$C_{\varphi}(\chi_{12} = 0, t) = \tfrac{1}{2}g(1+g), \tag{29}$$

$$\mathcal{N}_{\varphi}(\chi_{12} = 0, t) = \sqrt{x^2 - 4x + 1} + g - 1 \tag{30}$$

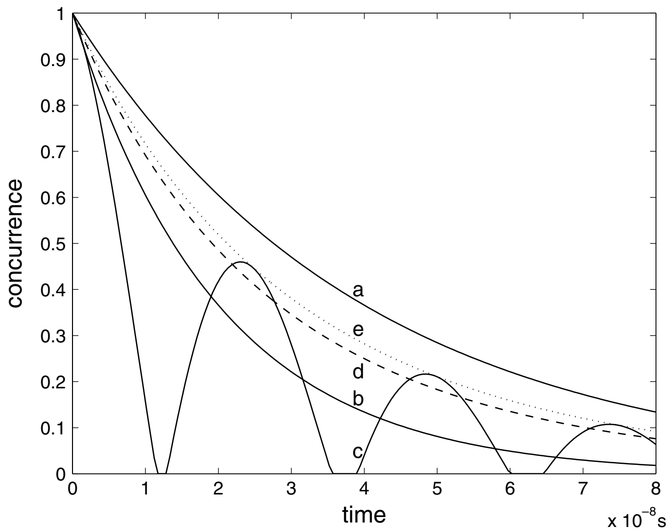

Figure 1. Decay of the concurrence for the initial Bell and Bell-like states: (*a*) $C_\psi(t)$, (*b*) $C_\phi(t)$, (*c*) $C_\varphi(\chi'_{12}, t)$, (*d*) $C_\varphi(\chi_{12} = 0, t)$ (dashed curve) and (*e*) $C_\varphi^{\mathrm{env}}(\chi'_{12}, t)$ (dotted curve) for cross-coupling constant $\chi'_{12} = 20$ rad MHz and damping constant $\gamma = 4$ rad MHz.

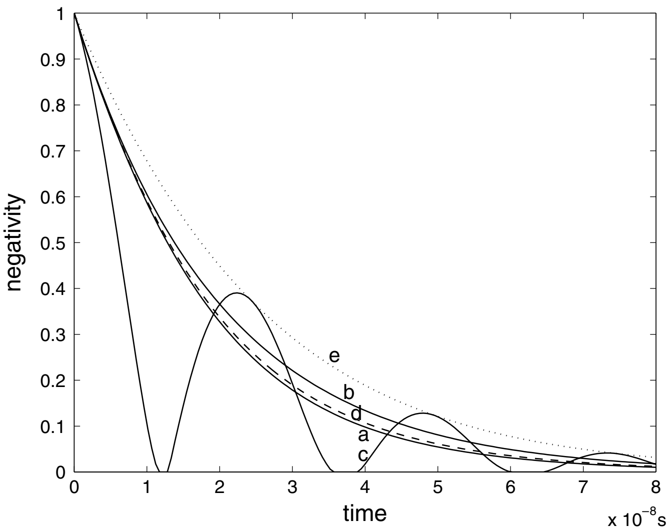

Figure 2. Decay of the negativity for the same Bell(-like) states and interactions as in figure 1: (*a*) $\mathcal{N}_\psi(t)$, (*b*) $\mathcal{N}_\phi(t)$, (*c*) $\mathcal{N}_\varphi(\chi'_{12}, t)$, (*d*) $\mathcal{N}_\varphi(\chi_{12} = 0, t)$ and (*e*) $\mathcal{N}_\varphi^{\mathrm{env}}(\chi'_{12}, t)$.

where $x = g(1-g)/2$. For nonzero damping and cross-coupling parameters, both the concurrence $C_\varphi(t)$ and the negativity $\mathcal{N}_\varphi(t)$ exhibit decaying oscillations, as shown by curves (*c*) in figures 1 and 2, respectively. The expressions for $C_\varphi(t)$ and $\mathcal{N}_\varphi(t)$ are quite lengthy in the general case of nonzero χ_{12} and γ, so we do not present them here. Instead we give approximate formulae for the envelope function of the concurrence

$$C_\varphi^{\mathrm{env}}(t) \approx \frac{g}{4}\left[\sqrt{x - \frac{2}{3}\left(z + \sqrt{2(x-2y)(x+y-z)}\right)} + g - 1\right] \tag{31}$$

where $x = 27 - 14g + 3g^2$, $y = \sqrt{159 - 129g + 37g^2 - 3g^3}$ and $z = \sqrt{(x + y)^2 - 9y^2}$, and for the envelope of the negativity

$$\mathcal{N}_\varphi^{\text{env}}(t) \approx \tfrac{1}{6}\left(2\,\text{Re}\sqrt[3]{v + \text{i}3(1 - g)g\sqrt{3w}} - (2 - g)^2 - g\right) \tag{32}$$

where $v = 8g^6 - 18g^5 - 93g^4 + 324g^3 - 273g^2 + 180g - 64$ and $w = 116g^6 - 316g^5 + 297g^4 + 930g^3 - 515g^2 + 624g + 16$. Equation (32) was derived by assuming only that the cross-coupling χ_{12} is much stronger than the damping constant γ, which implies that the function h in the density matrix (27) approaches unity. The envelope functions, given by (31) and (32), are depicted by curves (e) in figures 1 and 2, respectively. Another simpler but far less accurate approximation of the negativity envelope function can be given by

$$\mathcal{N}_\varphi^{\text{env}}(t) \approx \frac{1}{2}\sqrt{\frac{g^3(g^3 - 3g^2 - g + 11)}{g^2 - 3g + 4}} \tag{33}$$

which was obtained by using the general properties of the eigenvalues μ_i of the partial transpose $\hat{\rho}_\varphi^{\text{T}_1}(t)$ of density matrix (27), including $\sum_i \mu_i = 1$ and $\prod_i \mu_i = \det \hat{\rho}_\varphi^{\text{T}_1}(t)$, and observing that there exist two eigenvalues μ_i summing up approximately to zero. It is worth noting that the envelope functions, the same as those given by (31) and (32), are for the system initially in the separable state given by (18), for which $\hat{\rho}(t)$ has the form of (27) but with the functions h and f modified as follows: $h = [\gamma(2 + fg) - 2\text{i}\chi_{12}]/(\gamma - 2\text{i}\chi_{12})$ and $f = -\exp(2\text{i}\chi_{12}t)$. Note also that the envelope functions (31)–(33) are independent of the cross-coupling χ_{12} under assumption $\chi_{12} \gg \gamma$ but, even in this regime, the period of entanglement oscillations is a function of χ_{12}. A closer comparison of the entanglement for the Kerr interacting and non-interacting qubits in the lossy cavity leads us to the following inequalities $C_\varphi^{\text{env}}(t_n) = C_\varphi(\chi_{12} > 0, t_n) > C_\varphi(\chi_{12} = 0, t_n)$ and $\mathcal{N}_\varphi^{\text{env}}(t_n) = \mathcal{N}_\varphi(\chi_{12} > 0, t_n) > \mathcal{N}_\varphi(\chi_{12} = 0, t_n)$ valid for the moments of time equal to $t_n = n\pi/\chi_{12}$ for $n = 1, \ldots$. By comparing the entanglement measures for all the analysed MESs (see figures 1 and 2), we can finally conclude that

$$C_\psi(t) \geqslant C_\varphi^{\text{env}}(\chi_{12} \geqslant 0, t) \geqslant C_\varphi(\chi_{12} = 0, t) \geqslant C_\phi(t), \tag{34}$$

$$\mathcal{N}_\psi(t) \leqslant \mathcal{N}_\varphi(\chi_{12} = 0, t) \leqslant \mathcal{N}_\phi(t) \leqslant \mathcal{N}_\varphi^{\text{env}}(\chi_{12} \not\ll \gamma, t) \tag{35}$$

where the equalities hold for the nonzero damping constant γ at the evolution moments $t = 0$ and $t = \infty$, while for $\gamma = 0$ at any times t. Inequalities (34), (35), except those for $C_\varphi^{\text{env}}(\chi_{12} \geqslant 0, t)$ and $\mathcal{N}_\varphi^{\text{env}}(\chi_{12} \not\ll \gamma, t)$, can be proved analytically, while the remaining inequalities were checked numerically for a large class of parameters. Note that for small values of χ_{12} in comparison to γ it holds $\mathcal{N}_\varphi(\chi_{12}, t) \leqslant \mathcal{N}_\phi(t)$, nevertheless the last inequality in (35) is satisfied even if $\chi_{12} \sim \gamma$, and more pronounced for $\chi_{12} \gg \gamma$ (see figure 2), which is the condition assumed in the derivation of (31)–(33). Obviously, inequalities corresponding to (34) hold for the entanglement of formation, $E_F(t)$, and those corresponding to (35) are also valid for the PPT-entanglement cost, $E_N(t)$. The main conclusion is the following physical interpretation of inequalities (34), (35): by enabling Kerr interactions between the qubits initially in the Bell-like state, given by (26), the loss of the entanglement can be periodically reduced.

5. Decoherence of the maximally entangled mixed states

We will analyse the decoherence process of the initially maximally entangled mixed states of two qubits [32, 33, 35] on the example of the Werner states [36] defined to be

$(1/3 < p \leqslant 1)$:

$$\hat{\rho}_{p\psi\pm}(0) = p|\psi_\pm\rangle\langle\psi_\pm| + \frac{1-p}{4}\hat{I}_1 \otimes \hat{I}_2, \quad (36)$$

$$\hat{\rho}_{p\phi\pm}(0) = p|\phi_\pm\rangle\langle\phi_\pm| + \frac{1-p}{4}\hat{I}_1 \otimes \hat{I}_2, \quad (37)$$

where $|\psi_\pm\rangle$ and $|\phi_\pm\rangle$ are given by (19) and (23), respectively, and $\hat{I}_{1,2}$ are the identity 2×2 matrices. Thus, the Werner states are mixtures of a MES (Bell state) and the maximally mixed state, given by $\hat{I}_1 \otimes \hat{I}_2$, which can be interpreted as an equal incoherent mixture of the four Bell states. It is worth mentioning that the standard two-qubit Werner state is defined as $\hat{\rho}_{p\psi-}(0)$ only [36]. This state, given in terms of the singlet state $|\psi_-\rangle$, is invariant if both qubits are subjected to the same unitary transformation, $U \otimes U$. Nevertheless, by ignoring the $U \otimes U$ invariance but keeping the same entanglement properties, the standard Werner state is often generalized (see, e.g., [26, 33, 34]) to include mixtures of any MESs, as given by (36) and (37). Following this convention, we will apply the generalized definitions of Werner states in our study.

It is easy to show that the concurrences and negativities of the Werner states are the same and given by

$$C_{p\psi}(0) = C_{p\phi}(0) = \mathcal{N}_{p\psi}(0) = \mathcal{N}_{p\phi}(0) = (3p-1)/2. \quad (38)$$

The Werner states can be considered the MEMSs since their degree of entanglement cannot be increased by any unitary operations [32] and they have the maximum of entanglement for a given linear entropy (and vice versa) [33]. In a special case of $p = 1$, the Werner states go over into the MESs. The evolution of $\hat{\rho}_{p\psi}(t)$ for the initial Werner state (36) in the lossy nonlinear cavity is described by

$$\hat{\rho}_{p\psi\pm}(t) = \tfrac{1}{4}\{[(2-g)^2 - g^2 p]|00\rangle\langle 00| + g^2(1-p)|11\rangle\langle 11| \pm 2gp[\mathrm{e}^{\mathrm{i}(\chi_1-\chi_2)t}|01\rangle\langle 10| + \mathrm{e}^{-\mathrm{i}(\chi_1-\chi_2)t}|10\rangle\langle 01|] + g[2 - g(1-p)](|01\rangle\langle 01| + |10\rangle\langle 10|)\}, \quad (39)$$

being independent of the cross-coupling χ_{12}, which implies that a monotonical decrease of the entanglement occurs according to

$$C_{p\psi}(t) = \max\left\{0, gp - g\sqrt{(1-g)(1-p) + \frac{g^2(1-p)^2}{4}}\right\},$$
$$\mathcal{N}_{p\psi}(t) = \max\left\{0, \sqrt{(1-g)^2 + g^2p^2} - \frac{g^2(1-p)}{2} - (1-g)\right\}. \quad (40)$$

In a special case of $p = 1$, the above formulae simplify to (21) and (22), respectively. On the other hand, the evolution of $\hat{\rho}_{p\phi}(t)$ from the initial Werner state (37) reads

$$\hat{\rho}_{p\phi\pm}(t) = \tfrac{1}{2}\{(2 - 2g + x_p)|00\rangle\langle 00| + (g - x_p)(|01\rangle\langle 01| + |10\rangle\langle 10|) \pm p(f|00\rangle\langle 11| + f^*|11\rangle\langle 00|) + x_p|11\rangle\langle 11|\} \quad (41)$$

where $x_p = (1+p)g^2/2$ and $f = g\exp[\mathrm{i}(\chi_1 + 2\chi_{12} + \chi_2)t]$. Hence, the time evolution explicitly depends on the cross-coupling χ_{12}, but in such a way that the concurrence and negativity exhibit the same monotonic decrease independent of χ_{12} as follows:

$$C_{p\phi}(t) = \mathcal{N}_{p\phi}(t) = \max\left\{0, \frac{g}{2}[g(1+p) - 2(1-p)]\right\} \quad (42)$$

In a special case of $p = 1$, equation (42) reduces to g^2 in agreement with (25). Note that the subscript $\pm$ in $C_{p\psi}$, $N_{p\psi}$, $C_{p\phi}$ and $N_{p\phi}$ has been omitted as the functions are independent of the sign in (36) and (37).

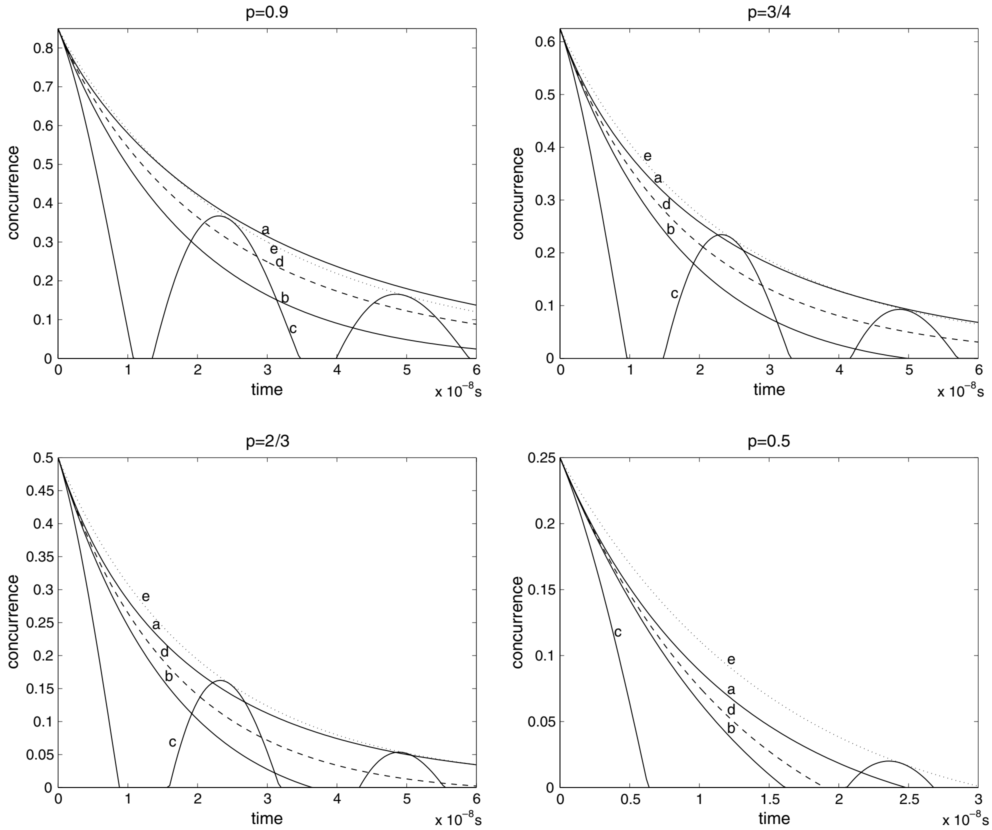

Figure 3. Decay of the concurrence for the initial Werner(-like) states: (*a*) $C_{p\psi}(t)$, (*b*) $C_{p\phi}(t)$, (*c*) $C_{p\varphi}(\chi'_{12}, t)$, (*d*) $C_{p\varphi}(\chi_{12} = 0, t)$ and (*e*) $C^{\text{env}}_{p\varphi}(\chi'_{12}, t)$ for various values of parameter p; γ and χ'_{12} are the same as in figure 1.

As the last example, let us assume qubits to be initially in the Werner-like state defined by $(1/3 < p \leqslant 1)$:

$$\hat{\rho}_{p\varphi}(0) = p|\varphi\rangle\langle\varphi| + \frac{1-p}{4}\hat{I}_1 \otimes \hat{I}_2 \tag{43}$$

in terms of the MES given by (26). The concurrence and negativity for (43) are equal to $C_{p\varphi}(0) = \mathcal{N}_{p\varphi}(0) = (3p-1)/2$ being the same as for the other Werner states. However, its evolution essentially differs from $\hat{\rho}_{p\psi\pm}(t)$ and $\hat{\rho}_{p\phi\pm}(t)$ by exhibiting oscillations of the entanglement. In detail, it is described by the density matrix elements

$$[\hat{\rho}_{p\varphi}(t)]_{ij} = p^{(1-\delta_{ij})}[\hat{\rho}_{\varphi}(t)]_{ij} \tag{44}$$

given in terms of (27), but with the off-diagonal terms multiplied by p as δ_{ij} stands for the Kronecker delta. In a special case of the lossless nonlinear cavity, the entanglement of the state $\hat{\rho}_{p\varphi}(\gamma = 0, t)$ evolves periodically as follows:

$$C_{p\varphi}(\gamma = 0, t) = \mathcal{N}_{p\varphi}(\gamma = 0, t) = \tfrac{1}{2}\max\{0, p(2|\cos(\chi_{12}t)| + 1) - 1\} \tag{45}$$

which is opposite to the time-independent evolution of the other Werner states, namely $\hat{\rho}_{p\psi\pm}(\gamma = 0, t) = \hat{\rho}_{p\phi\pm}(\gamma = 0, t) = \text{const}$. One can conclude from (45) (see also figures 3 and 4) that by decreasing parameter p, the entanglement and the time intervals in

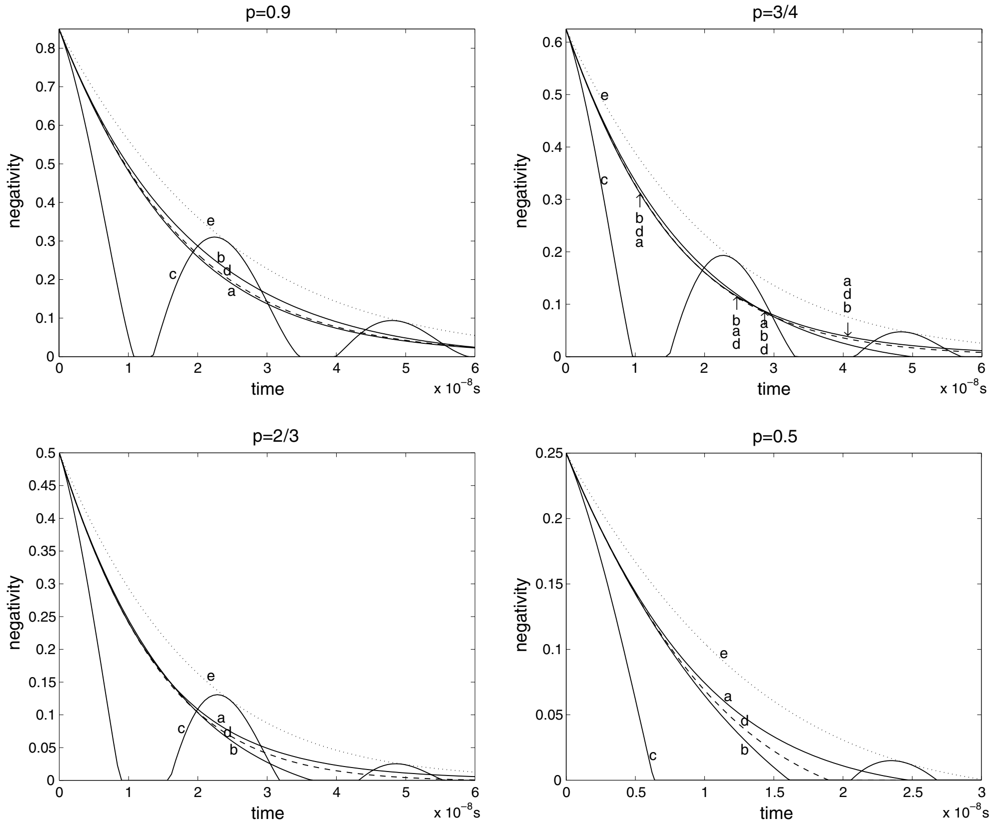

Figure 4. Decay of the negativity for the same Werner(-like) states and interactions as in figure 3: (*a*) $\mathcal{N}_{p\psi}(t)$, (*b*) $\mathcal{N}_{p\phi}(t)$, (*c*) $\mathcal{N}_{p\varphi}(\chi'_{12}, t)$, (*d*) $\mathcal{N}_{p\varphi}(\chi_{12} = 0, t)$ and (*e*) $\mathcal{N}^{\rm env}_{p\varphi}(\chi'_{12}, t)$.

which the states are entangled decrease. For the dissipative nonlinear cavity, the entanglement corresponding to the evolution of $\hat{\rho}_{p\varphi}(t)$ exhibits decaying oscillations shown by curves (*c*) in figures 3 and 4. As in the previous section, we are mainly interested in the envelopes of these oscillations. In the special case of $p = 1$, when the initial Werner-like state goes over into the Bell-like state, the concurrence and negativity envelopes are given by (31) and (32), respectively. By assuming $\chi_{12} \gg \gamma$, an approximate formula for the p-dependent envelopes of the concurrence can be given by

$$C^{\rm env}_{p\varphi}(t) \approx \frac{g}{4} \max \left\{0, \frac{1}{\sqrt{3}} \left(\sqrt{x_p + 4p\sqrt{y_p}} - 2\sqrt{x_p - 2p\sqrt{y_p}}\right) + g + p - 2\right\} \tag{46}$$

in terms of $x_p = 3G^2 + 2Gp + 11p^2$ and $y_p = 3G^3 + G^2(10 + 9p) + G(3 + 14p) + p(9 + 16p)$, where $G = 2 - g$. Note that (46) for $p = 1$ is another approximate formula of the concurrence envelope for the initial MES $|\varphi\rangle$, but leading to a slightly worse approximation than that given by (31). For brevity, the lengthy formula for the p-dependent negativity envelope, $\mathcal{N}^{\rm env}_{p\varphi}(t)$, generalizing equation (32), is not presented explicitly here although it was used for plotting the envelope curves (*c*) in figure 4. By analysing figures 3 and 4, we conclude that

$$\begin{aligned} C^{\rm env}_{p\varphi}(t_n) &= C_{p\varphi}(\chi_{12} > 0, t_n) \geqslant C_{p\varphi}(\chi_{12} = 0, t_n), \\ \mathcal{N}^{\rm env}_{p\varphi}(t_n) &= \mathcal{N}_{p\varphi}(\chi_{12} > 0, t_n) \geqslant \mathcal{N}_{p\varphi}(\chi_{12} = 0, t_n) \end{aligned} \tag{47}$$

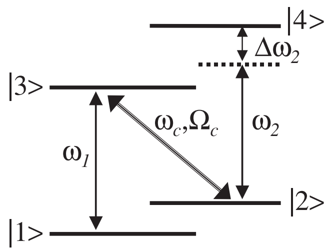

Figure 5. Four-level atomic system for the resonantly enhanced Kerr nonlinearity in the Schmidt–Imamoğlu scheme.

at moments of time $t_n \approx n\pi/\chi_{12}$ ($n = 1, 2, \ldots$), which means the decay of entanglement of the initially Werner-like state (43) in a lossy cavity can be periodically retarded by inserting the Kerr nonlinearity in the cavity.

6. Discussion and conclusions

Let us finally address the question whether the interactions studied in this paper can be experimentally observable. As mentioned in the introduction, the conditions assumed in the paper of the strong Kerr interaction at low light intensities can be satisfied, e.g., for the EIT schemes as studied theoretically [11, 13–17] and confirmed experimentally [12, 18]. Schmidt and Imamoğlu [11] have proposed a renowned EIT scheme where a low-density cloud of cold atoms with the four-level structure, shown in figure 5, exhibits giant resonantly enhanced nonlinear cross-coupling with vanishing linear susceptibilities at low intensities. In the scheme, atoms are placed in a cavity (or double cavity) tuned to two frequencies: ω_1 of the mode a_1 resonant with the transition $|1\rangle \leftrightarrow |3\rangle$, and ω_2 of the mode a_2 detuned by $\Delta\omega_2$ of the transition $|2\rangle \leftrightarrow |4\rangle$. The EIT effect for the modes a_1 and a_2 is induced by a classical coupling field of frequency ω_c resonant with the transition $|2\rangle \leftrightarrow |3\rangle$. By assuming that $|\omega_1 - \omega_2| \gg \Delta\omega_2$, no nonlinear self-coupling will occur in the system. Nevertheless, as shown in the preceding sections, only the cross-coupling changes the entanglement evolution. By adiabatically eliminating all the atomic levels, Schmidt and Imamoğlu have found the real part of the resonantly enhanced third-order nonlinear susceptibility, $\mathrm{Re}(\chi^{(3)})$, to be given by $|\mu_{13}|^2|\mu_{24}|^2 n_{\mathrm{at}}\left(2\epsilon_0\hbar^3\Omega_c^2\Delta\omega_2 V_{\mathrm{cav}}\right)^{-1}$, where μ_{ij} is the electric dipole matrix element between the states $|i\rangle$ and $|j\rangle$, n_{at} is the total number of atoms contained in the cavity of volume V_{cav}, Ω_c is the coupling-field Rabi frequency and ϵ_0 is the permittivity of free space. With the help of the expression for $\mathrm{Re}(\chi^{(3)})$, it is easy to show that the Kerr nonlinear cross-coupling is given by [13, 15]

$$2\chi_{12} \sim \frac{3|g_{13}|^2|g_{24}|^2}{\Omega_c^2\,\Delta\omega_2} n_{\mathrm{at}} \tag{48}$$

where $g_{ij} = \mu_{ij}\sqrt{\omega_i/(2\epsilon\hbar V_{\mathrm{cav}})}$ is the coupling coefficient between the atoms and the cavity mode a_i of frequency ω_i. It is worth stressing that the above formulae for $\mathrm{Re}(\chi^{(3)})$ and χ_{12} are valid under the condition that $|g_{13}|^2 n_{\mathrm{at}}/\Omega_c^2 < 1$ required by the applied adiabatic elimination procedure [37]. The EIT-enhanced Kerr-coupling constants for the Schmidt–Imamoğlu scheme can be estimated moderately as $\sim$0.2 rad MHz [15], or by putting the stringent limit on the required cavity parameters [37], as $\sim$100 rad MHz [13]. In our numerical analysis, we have chosen $\chi_{12} = 20$ rad MHz. A typical cavity decay rate obtainable in current experiments is of the order $\sim$4 rad MHz, which is five times smaller than the value of χ_{12} chosen for plotting figures 1–4. This estimation is less stringent than that given in [13]. It is worth noting that the EIT leads to remarkable light-speed reduction [12], which enables reduction of the cavity decay rate in the Schmidt–Imamoğlu setup with the same finesse mirrors. We have analysed decays within times <80 ns, which are fairly shorter than the dephasing time ($\sim$9 μs) for an atom cloud measured in the Hau *et al* experiment [12] but longer, for obvious reasons, than the evolution times ($\sim$8 ns) in the quantum non-demolition scheme of Duan *et al* [15].

In conclusion, we have analysed the evolution of two optical modes in qubit states interacting via a Kerr nonlinearity in a lossy cavity modelled by dissipative coupled nonlinear oscillators being initially in the maximally entangled pure or mixed states. We have found that for the initial Bell ($|\psi_\pm\rangle$ and $|\phi_\pm\rangle$) or Bell-like ($|\varphi\rangle$) states, the decay of the concurrence, or equivalently the entanglement of formation, is the slowest for $|\psi_\pm\rangle$ and the fastest for $|\phi_\pm\rangle$, while the decay of the negativity, or equivalently the PPT-entanglement cost, is the slowest for $|\varphi\rangle$ (if the nonlinearity parameter is much greater than the damping constants) and the fastest for $|\psi_\pm\rangle$. Thus, we have provided simple analytical examples of states differently ordered by concurrence and negativity. These seemingly inconsistent results are physically meaningful as discussed in, e.g., [7, 24, 26] and proved in general terms by Virmani and Plenio [27]. Nevertheless, to our knowledge, our analysis is the first demonstration of the relativity of the entanglement measures as a result of a physical process. Moreover, we have also studied decoherence of the initial maximally entangled mixed states on the example of three kinds of Werner(-like) states as related to the different Bell(-like) states $|\psi_\pm\rangle$, $|\phi_\pm\rangle$ and $|\varphi\rangle$. Our analytical and numerical results show the differences and similarities of the negativity and concurrence decays of the Werner(-like) states in comparison to the Bell(-like) states.

We have demonstrated that by inserting a medium with the Kerr nonlinearity, described by Hamiltonian (10), into the lossy cavity, evolution of the initial Bell states $|\psi_\pm\rangle$ or $|\phi_\pm\rangle$ and the corresponding Werner states is changed but in such a way that the entanglement decays in the same manner as without the nonlinear medium. However, if the qubits placed in a lossy cavity are initially in the Bell-like state $|\varphi\rangle$ or the corresponding Werner-like state, the loss of entanglement can be periodically delayed (partially recovered) by inserting a medium with the Kerr nonlinearity.

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