Optical-state truncation and teleportation of qudits by conditional eight-port interferometry

Adam Miranowicz

Nonlinear Optics Division, Physics Institute, Adam Mickiewicz University, 61-614 Poznań, Poland
and
SORST Research Team for Interacting Career Electronics, Graduate School for Engineering Science, Osaka University, Osaka 560-8531, Japan

Abstract

The quantum scissors device of Pegg *et al* (1998 *Phys. Rev. Lett.* **81** 1604) enables truncation of the Fock-state expansion of an input optical field to qubit and qutrit (three-dimensional) states only. Here, a generalized scissors device is proposed using an eight-port optical interferometer. Upon post-selection based on photon counting results, the interferometer implements generation and teleportation of qudit (d-dimensional) states by truncation of an input field at the $(d-1)$th term of its Fock-state expansion up to $d = 6$. Examples of selective truncations, which can be interpreted as a Fock-state filtering and hole burning in the Fock space of an input optical field, are discussed. Deterioration of the truncation due to imperfect photon counting is discussed including inefficiency, dark counts and realistic photon-number resolutions of photodetectors.

Keywords: linear optics, quantum state engineering, quantum teleportation, projection synthesis, multiport interferometer, quantum scissors, positive operator valued measure

1. Introduction

Quantum engineering of nonclassical light has attracted remarkable interest in the last decade [1]. This interest has been further stimulated by a recent theoretical demonstration of Knill *et al* [2] that linear optical systems enable efficient quantum computation. Such systems are experimentally realizable with present-day technology [3, 4], since they are based only on beam splitters (BSs) and phase shifters (PSs) together with photodetectors and single-photon sources.

In this paper, linear systems are studied for the *optical-state truncation*, which refers to truncation of the Fock-state expansion of an input optical state

$$|\psi\rangle = \sum_{n=0}^{\infty} \gamma_n |n\rangle, \qquad (1)$$

with unknown superposition coefficients γ_n, into the following finite superposition of d states:

$$|\phi_{\text{trun}}^{(d)}\rangle = \mathcal{N} \sum_{n=0}^{d-1} \gamma_n |n\rangle, \qquad (2)$$

which is called the optical *qudit* state (d-dimensional generalized qubit). Here $\mathcal{N} = (\sum_{n=0}^{d-1} |\gamma_n|^2)^{-1/2}$ is the renormalization constant. In the following, the similarity sign will be used instead of writing explicitly $\mathcal{N}$. The input state $|\psi\rangle$ can be a coherent state $|\alpha\rangle$, or any other infinite or finite-dimensional state. Systems for the optical-state truncation are referred to as *quantum scissors devices* (QSDs).

The first and simplest QSD was proposed by Pegg *et al* [5, 6], then analysed theoretically in various contexts by others [7–14], and experimentally realized by Babichev *et al* [15] and Resch *et al* [16]. This device, schematically depicted in figure 1, is composed of linear optical elements (two beam splitters BS1 and BS2) and photodetectors D_2 and D_4 (label 4 is used for consistency with the other schemes that are discussed in the following). If a single-photon Fock state $|1\rangle$ is in one of
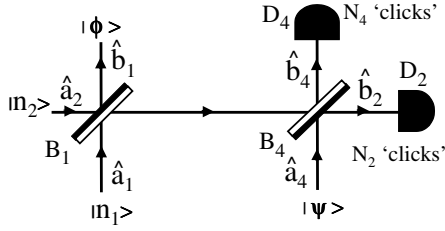
Figure 1. Six-port quantum scissors device of Pegg, Phillips and Barnett. Key: $|\psi\rangle$—input state which, in particular, can be a coherent state $|\alpha\rangle$; $|n_j\rangle$—input Fock states; $|\phi\rangle$—output qubit or qutrit state; D_j—photon counters; B_j—beam splitters; $\hat{a}_j$ and $\hat{b}_j$—input and output annihilation operators, respectively.

the input modes $\hat{a}_1$ or $\hat{a}_2$ and the vacuum state is in the other input, while the measurement has resulted in a single count in one of the detectors and no count in the other, then the state $|\psi\rangle$, given by (1), in the input mode $\hat{a}_4$ can be truncated to the qubit state

$$|\phi_{\text{trun}}^{(2)}\rangle \sim \gamma_0|0\rangle + \gamma_1|1\rangle \qquad (3)$$

in the output mode $\hat{b}_1$, as a special case of (2). Detailed analysis of the conditions to obtain the target state (3) as a function of the BS parameters, input states and measurement outcomes will be reviewed in section 3. The described optical state truncation based on conditional measurements is referred to as the *projection synthesis* [5], which is a powerful method applied also for other purposes [17–21]. By analysing figure 1, it is easy to observe that light from the input mode $\hat{a}_4$ cannot reach the output mode $\hat{b}_1$. So, one can conclude that the truncation is achieved via quantum teleportation (see [22], for a review see [23]), though not of the entire input state but only of the first two terms of its Fock-state expansion [5, 7, 8, 15, 24]. Thus, in a special case, if the input field is already prepared in a qubit state, then the scissors become a conventional teleporting device. However, contrary to the unconditional teleportation scheme of Bennett *et al* [22], the teleportation (and truncation) via the Pegg–Phillips–Barnett QSD is successful only in the cases when the two detectors count one photon in total.

The concept of optical-state truncation is by no means limited to the discussed truncation of the number-state expansion of a given state. For instance, by considering truncation of a coherent state, defined by the action of the displacement operator on the vacuum state, one can truncate the displacement operator and then apply it to the vacuum state. Such a truncated state is essentially different (for $d > 2$) from that given by (2) [25–27]. Nevertheless, it is physically realizable, for example, in a pumped ring cavity with a Kerr nonlinear medium as was demonstrated by Leoński *et al* [28–31]. The QSD schemes can also be generalized for the truncation of two-mode [32] or multi-mode fields.

The original Pegg–Phillips–Barnett QSD enables the truncation of an input state only to qubit and qutrit (three-dimensional qudit) states, as was shown by Koniorczyk *et al* [8]. Here, we discuss a generalization of the QSD for truncation and teleportation of $d = 2, \ldots, 6$-dimensional qudits and suggest a way to extend the scheme for an arbitrary d. Our approach is essentially different from the other qudit truncation schemes [8, 11, 29, 30] and we believe that it is easier to be experimentally realized. The proposed scheme is
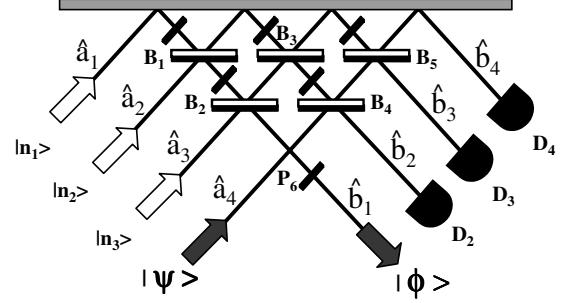
Figure 2. Generalized eight-port quantum scissors device. The notation is the same as in figure 1 but $|\phi\rangle$ denotes the output qudit state; phase shifters P_6 and P_j, in front of the beam splitter B_j, are shown by small black bars, while the mirror is depicted by a large bar.

based on an eight-port optical interferometer shown in figure 2. The setup resembles a well-known multiport interferometer of Zeilinger *et al* [33, 34], which has been theoretically analysed [35–39] and experimentally applied [40–42] for various purposes but, to our knowledge, has not yet been used for optical-state truncation. An important difference between the standard multiport and that proposed here lies in the elimination of the apex BS of the triangle. This elimination is important for the processes of truncation and teleportation.

The paper is organized as follows. In section 2, the generalized quantum scissors device is proposed including a description of the setup (2.1), a short review of multiport unitary transformations (2.2), and an explanation of the projection synthesis (2.3), which enables the qudit state truncation. Reductions of the generalized QSD to the Pegg–Phillips–Barnett QSD are demonstrated in section 3. Detailed analyses of the generalized QSD for the truncation up to three, four and five photon-number states are given in sections 4, 6, and 7, respectively. Selective truncations, which can be interpreted as a Fock-state filtering [43] or a hole burning in Fock space [44, 45], are discussed in section 5. How imperfect photon counting deteriorates the truncation processes is discussed in section 8 by including realistic photon-number resolutions, inefficiency, and dark counts of photodetectors. Final conclusions with a discussion of open problems including a generalization of the scheme for truncations of an arbitrary qudit state are presented in section 9.

2. Generalized QSD

2.1. The setup

We analyse a generalized quantum scissors device (GQSD) based on an eight-port optical interferometer, also referred to as a multiport mixer or multiport beam splitter, which is assembled in a pyramid-like configuration of ordinary beam splitters (BSs) and phase shifters (PSs) as shown in figure 2. The most general four-port beam-splitter scattering matrix reads as (see, e.g., [46, 47])

$$\mathbf{B}' = e^{i\theta_0} \begin{bmatrix} t\exp(i\theta_t) & r\exp(i\theta_r) \\ -r\exp(-i\theta_r) & t\exp(-i\theta_t) \end{bmatrix} \qquad (4)$$

where $T = t^2$ describes the transmittance, and $R = r^2 = 1-T$ is the reflectance of the BS. The associated phase factors θ_t and

θ_r can be realized by the external phase shifters, described by $\mathbf{P}_\pm = \mathrm{diag}[\exp(\mathrm{i}\theta_t \pm \mathrm{i}\theta_r), 1]$, which are placed in front of and behind the beam splitter described by real scattering matrix

$$\mathbf{B} = \begin{bmatrix} t & r \\ -r & t \end{bmatrix} \qquad (5)$$

as comes from the decomposition

$$\mathbf{B}' = \exp(\mathrm{i}\theta_0')\mathbf{P}_+\mathbf{B}\mathbf{P}_- \qquad (6)$$

where $\theta_0' = \theta_0 - \theta_t$. Without loss of generality the global phase factors $\exp(\mathrm{i}\theta_0')$ and $\exp(\mathrm{i}\theta_0)$ can be omitted. Note that (5) describes an asymmetric BS, as marked in all figures by bars with distinct surfaces. We use the same convention as in [48, 49] that beams reflected from the white surface are π phase shifted so, according to (5), the reflection from the black surface and transmissions from any side are without phase shift.

The total scattering matrix $\mathbf{S}$ of the GQSD, shown in figure 2, can be given by

$$\mathbf{S} = \mathbf{P}_6\mathbf{B}_5\mathbf{P}_5\mathbf{B}_4\mathbf{P}_4\mathbf{B}_3\mathbf{P}_3\mathbf{B}_2\mathbf{P}_2\mathbf{B}_1\mathbf{P}_1 \qquad (7)$$

which is the sequence of two-mode 'real' beam splitters described explicitly by

$$\mathbf{B}_1 = \begin{bmatrix} t_1 & r_1 & 0 & 0 \\ -r_1 & t_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B}_2 = \begin{bmatrix} t_2 & 0 & r_2 & 0 \\ 0 & 1 & 0 & 0 \\ -r_2 & 0 & t_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{B}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t_3 & r_3 & 0 \\ 0 & -r_3 & t_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{B}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t_4 & 0 & r_4 \\ 0 & 0 & 1 & 0 \\ 0 & -r_4 & 0 & t_4 \end{bmatrix},$$

$$\mathbf{B}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t_5 & r_5 \\ 0 & 0 & -r_5 & t_5 \end{bmatrix} \qquad (8)$$

and phase shifters represented by

$$\mathbf{P}_k = \mathrm{diag}[\exp(\mathrm{i}\xi_k), 1, 1, 1] \qquad \text{for } k = 1, 2, 6,$$
$$\mathbf{P}_k = \mathrm{diag}[1, \exp(\mathrm{i}\xi_k), 1, 1] \qquad \text{for } k = 3, 4, \qquad (9)$$
$$\mathbf{P}_5 = \mathrm{diag}[1, 1, \exp(\mathrm{i}\xi_5), 1].$$

Similarly, the diagonal matrix

$$\mathbf{M}_k = \mathrm{diag}[\exp(\mathrm{i}\zeta\,\delta_{1k}), \exp(\mathrm{i}\zeta\,\delta_{2k}), \exp(\mathrm{i}\zeta\,\delta_{3k}), \exp(\mathrm{i}\zeta\,\delta_{4k})] \qquad (10)$$

describes the kth mode reflection phase shift caused by the mirror (see figure 2), where δ_{jk} is the Kronecker delta. We have not explicitly written $\mathbf{M}_k$ in the sequence (7), since the reflection phase shifts ζ for modes 1, 2 and 3 can be incorporated in ξ_1, ξ_3 and ξ_5, respectively. Besides, the reflection phase shift in mode 4 does not affect photodetection in D_4, and thus can be neglected.

The described setup resembles a well-known multiport interferometer in triangle configuration of the beam splitters as studied in various contexts since its theoretical proposal and experimental realizations by Zeilinger *et al* [33–35, 40]. However, an important difference between the standard multiport interferometer and that analysed here is the elimination of the apex BS of the triangle, which is usually placed at the crossing of beams 1 and 4 in figure 2. In particular, the setup for the Pegg–Phillips–Barnett QSD resembles the Zeilinger six-port interferometer with one BS removed. This elimination is essential for the processes of truncation and teleportation of qudits as will be shown in the following.

2.2. Multiport unitary transformation

The annihilation operators $\hat{a}_i$ at the N inputs to multiport linear interferometer are related to the annihilation operators $\hat{b}_i$ at the N outputs as follows (see, e.g., [47, 49]):

$$\hat{b}_i = \hat{U}^\dagger \hat{a}_i \hat{U} = \sum_{j=1}^{N} S_{ij}\hat{a}_j \qquad (11)$$

where S_{ij} are the elements of the unitary scattering matrix $\mathbf{S}$, and $\hat{U}$ is the unitary operator describing the evolution of the N-mode input state, say $|\Psi\rangle$, into the N-mode output state, say $|\Phi\rangle$:

$$|\Phi\rangle = \hat{U}|\Psi\rangle. \qquad (12)$$

By introducing the column vectors $\hat{\mathbf{a}} \equiv [\hat{a}_1; \hat{a}_2; \ldots; \hat{a}_N]$ and $\hat{\mathbf{b}} \equiv [\hat{b}_1; \hat{b}_2; \ldots; \hat{b}_N]$, the set of equations (11) can compactly be rewritten as

$$\hat{\mathbf{b}} = \hat{U}^\dagger \hat{\mathbf{a}} \hat{U} = \mathbf{S}\hat{\mathbf{a}} \qquad (13)$$

from which follow the inverse relations for the creation operators

$$\hat{\mathbf{a}}^\dagger = \hat{U}\hat{\mathbf{b}}^\dagger \hat{U}^\dagger = \mathbf{S}^T\hat{\mathbf{b}}^\dagger. \qquad (14)$$

Then, one can observe that

$$\hat{U}\hat{a}_i^\dagger \hat{U}^\dagger = \hat{U}\Big(\sum_j S_{ji}\hat{b}_j^\dagger\Big)\hat{U}^\dagger = \sum_j S_{ji}\hat{U}\hat{b}_j^\dagger \hat{U}^\dagger = \sum_j S_{ji}\hat{a}_j^\dagger. \qquad (15)$$

By applying (15) and noting that neither BSs nor PSs change the vacuum state, $\hat{U}|\mathbf{0}\rangle = |\mathbf{0}\rangle$, one can calculate the output state $|\Phi\rangle$ of the multiport interferometer described by the scattering matrix $\mathbf{S}$ for the input Fock states $|\Psi\rangle = |n_1, \ldots, n_N\rangle \equiv |\mathbf{n}\rangle$ as follows (for detailed examples see [47, 49]):

$$\hat{U}|\mathbf{n}\rangle = \hat{U}\prod_{i=1}^{N}\frac{(\hat{a}_i^\dagger)^{n_i}}{\sqrt{n_i!}}|\mathbf{0}\rangle$$

$$= \prod_{i=1}^{N}\frac{1}{\sqrt{n_i!}}(\hat{U}\hat{a}_i^\dagger \hat{U}^\dagger)^{n_i}|\mathbf{0}\rangle$$

$$= \prod_{i=1}^{N}\frac{1}{\sqrt{n_i!}}\Big(\sum_{j=1}^{N} S_{ji}\hat{a}_j^\dagger\Big)^{n_i}|\mathbf{0}\rangle$$

$$= \frac{1}{\sqrt{n_1!\cdots n_N!}}\sum_{\mathbf{j}=1}^{N}\prod_{l=1}^{M} S_{j_l x_l}\hat{a}_{j_l}^\dagger|\mathbf{0}\rangle \qquad (16)$$

where $M = \sum_i n_i$ is the total number of photons; $\{x_j\} \equiv (\underbrace{1, \ldots, 1}_{n_1}, \underbrace{2, \ldots, 2}_{n_2}, \ldots, \underbrace{N, \ldots, N}_{n_N})$ labelled by $l = 1, \ldots, M$; $\sum_{\mathbf{j}}$ stands for the multiple sum over $j_1, j_2, \ldots, j_M$. We apply the last equation of (16) in our analytical approach, and the second last equation in our numerical analysis.

(a)

(b)
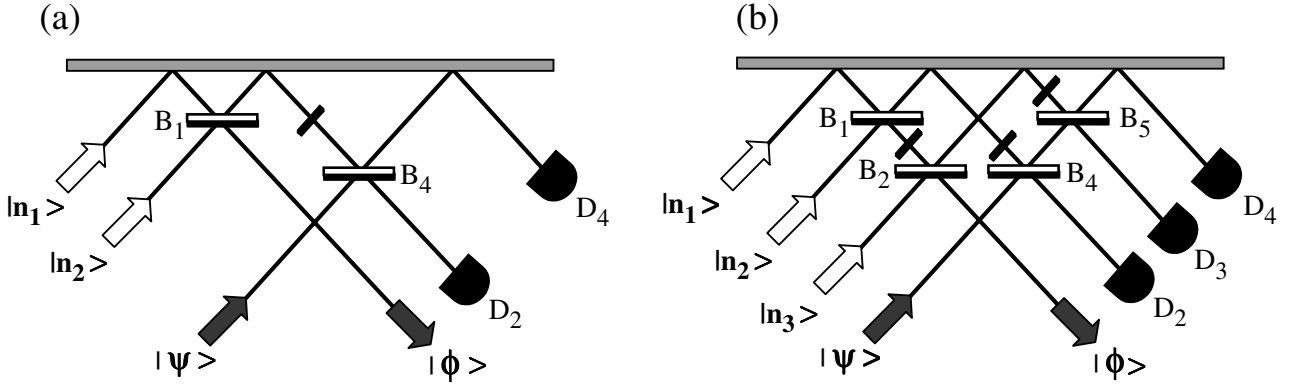
Figure 3. Special cases of the GQSD, shown in figure 2, corresponding to (a) the Pegg–Phillips–Barnett QSD in figure 1, and (b) the scheme analysed in sections 4 and 6.

2.3. Projection synthesis and teleportation

We are interested in optical truncation schemes based on the generalized QSD, shown in figure 2, or its simplified versions depicted in figures 3(a), (b). The input mode $\hat{a}_4$, being in an arbitrary pure state, given by (1), is truncated to a qudit state, given by (2), in mode $\hat{b}_1$. To achieve the truncation desired, we assume that modes $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ are in the Fock states $|n_1\rangle$, $|n_2\rangle$, and $|n_3\rangle$, respectively. Thus, the total four-mode input state is

$$|\Psi\rangle = |n_1\rangle_1 |n_2\rangle_2 |n_3\rangle_3 |\psi\rangle_4 \equiv |n_1 n_2 n_3 \psi\rangle \qquad (17)$$

which is transformed into the output state $|\Phi\rangle$, according to (12). Now, photon-counting of the output modes $\hat{b}_2$, $\hat{b}_3$, and $\hat{b}_4$ is performed yielding N_2, N_3 and N_4 photons, respectively. If the total number of detected photons is equal to the sum of photons in modes $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, i.e., $N_2 + N_3 + N_4 = n_1 + n_2 + n_3 \equiv d - 1$, then the total four-mode output state $|\Phi\rangle$ is reduced to the following single-mode state:

$$|\phi\rangle \equiv |\phi_{n_1 n_2 n_3}^{N_2 N_3 N_4}\rangle = \mathcal{N}\,_2\langle N_2|\,_3\langle N_3|\,_4\langle N_4|\Phi\rangle$$
$$= \mathcal{N} \sum_{n=0}^{d-1} \langle n N_2 N_3 N_4 |\Phi\rangle |n\rangle = \mathcal{N} \sum_{n=0}^{d-1} c_n^{(d)} \gamma_n |n\rangle \qquad (18)$$

with the amplitudes $c_n^{(d)}$ defined as

$$c_n^{(d)}(\mathbf{T}, \boldsymbol{\xi}) = \langle n N_2 N_3 N_4 | \hat{U} | n_1 n_2 n_3 n \rangle \qquad (19)$$

depending, in particular, on the beam splitter transmittances $\mathbf{T} \equiv [t_1^2, t_2^2, t_3^2, t_4^2, t_5^2]$ and phase shifts $\boldsymbol{\xi} \equiv [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5]$ and ξ_6. In the following, we present solutions for $c_n^{(d)}$ assuming $\xi_6 = 0$. Nevertheless, since the action of the phase shifter P_6 with ξ_6 simply corresponds to the transformation of $c_n^{(d)}$ into $c_n^{(d)} \exp(n\xi_6)$, one can readily obtain the general solutions for any ξ_6.

Perfect truncation is achieved independently of the form of the input state $|\psi\rangle$ if the amplitudes $c_n^{(d)}$ are equal to each other for all $n \leqslant d - 1$ or, equivalently,

$$\Delta(\mathbf{T}, \boldsymbol{\xi}) \equiv \sum_{n=1}^{d-1} |c_n^{(d)}(\mathbf{T}, \boldsymbol{\xi}) - c_0^{(d)}(\mathbf{T}, \boldsymbol{\xi})| = 0 \qquad (20)$$

for some properly chosen values of the BS transmittances $\mathbf{T}$ and phase shifts $\boldsymbol{\xi}$. So, the problem is to find such $\mathbf{T}$ and $\boldsymbol{\xi}$, for which

the amplitudes $c_n^{(d)}$ satisfy condition (20). It is worth noting that, for a possible experimental realization of the scheme, it is essential to have BSs with variable transmittance. A possible solution is to replace each of the BSs by a Mach–Zehnder interferometer composed of two symmetric 50:50 BSs, two mirrors and two PSs [9, 35, 41].

Our numerical minimalization of Δ reveals that perfect truncation using the generalized QSD can be realized for various values of the BS transmittances and phase shifts, which, however, correspond to different probabilities of successful truncation. Here, we focus on very simple analytical solutions rather than numerically optimized approximate ones. In the following, we will analyse in detail truncations up to six-dimensional qudits.

The described truncation process via the GQSD can be considered as a kind of form-limited quantum teleportation of the first d terms of the Fock-state expansion of the incident light in analogy to the qutrit-limited teleportation via the Pegg–Phillips–Barnett QSD discussed in [5, 7, 8, 15, 23]. Even a brief analysis of figure 2 shows that no light from input port 4 can reach output port 1. In fact, this transformation is based on the same principles of quantum entanglement and Bell-state measurement (the projection postulate) as the original Bennett *et al* teleportation scheme [22]. The multi-photon entangled state is created by the beam splitters and, as required, the original state $|\psi\rangle$ is destroyed by the Bell-state measurement implemented by BSs 3–5 and detectors 2–4. Thus, by assuming that an incident light is already prepared in a d-dimensional (up to $d = 6$) qudit state $|\psi^{(d)}\rangle$ and the conditional measurement is successfully performed, then the state is teleported from mode $\hat{a}_4$ to $|\phi^{(d)}\rangle = |\psi^{(d)}\rangle$ in mode $\hat{b}_1$.

3. Reductions to the Pegg–Phillips–Barnett QSD

First, we show how the eight-port QSD can truncate the input state $|\psi\rangle$, given by (1), to a qubit state, given by (3), as expected by the original Pegg–Phillips–Barnett scissors device [5, 6]. The system shown in figure 1 is a special case of that shown in figure 2 by assuming, for example, that BSs 1, 3 and 5 are perfectly reflecting, thus a complete set of transmittances is $\mathbf{T} = [0, t_2^2, 0, t_4^2, 0]$. Note that, in this configuration, input port 1 and output port 4 are unimportant, and so can be neglected. Another way to generate the truncated

state (3) is to remove BSs 2, 3 and 5 as shown in figure 3(a). Thus, the simplified version of the GQSD is described by the transmittances

$$\mathbf{T} = [t_1^2, 1, 1, t_4^2, 1], \qquad (21)$$

and we can also set that the only nonzero phase shift is ξ_4. In this configuration, the input mode $\hat{a}_3$ is only reflected from the large mirror and leaves the system without interfering with the other modes. Thus, the truncation occurs independently of the input state $|n_3\rangle$ and the photon counting result in detector D_3. Let $|\phi_{n_1 n_2}^{N_2 N_4}\rangle$ denote the output state $|\phi\rangle$ in mode $\hat{b}_1$ for the input states $|n_1\rangle$, $|n_2\rangle$, $|\psi\rangle$ in modes $\hat{a}_1, \hat{a}_2, \hat{a}_4$, together with detection of N_2 and N_4 photons in detectors D_2 and D_4, respectively. Note that equivalently one can analyse $\mathbf{T} = [1, t_2^2, 1, 1, t_5^2]$ instead of (21) to reduce the system to the Pegg–Phillips–Barnett QSD. By applying (7) to (16) with transmittances given by (21), one readily finds the output states:

$$
\begin{aligned}
|\phi_{10}^{01}\rangle &\sim e^{i\xi_4} r_1 r_4 \gamma_0 |0\rangle + t_1 t_4 \gamma_1 |1\rangle, \\
|\phi_{01}^{10}\rangle &\sim e^{i\xi_4} t_1 t_4 \gamma_0 |0\rangle + r_1 r_4 \gamma_1 |1\rangle, \\
|\phi_{01}^{01}\rangle &\sim -e^{i\xi_4} t_1 r_4 \gamma_0 |0\rangle + r_1 t_4 \gamma_1 |1\rangle, \\
|\phi_{10}^{10}\rangle &\sim -e^{i\xi_4} r_1 t_4 \gamma_0 |0\rangle + t_1 r_4 \gamma_1 |1\rangle.
\end{aligned}
\qquad (22)
$$

As follows from (22), the states $|\phi_{10}^{01}\rangle$ and $|\phi_{01}^{10}\rangle$ become perfectly truncated qubit states if $t_1 = r_4$ and $\xi_4 = 0$. On the other hand, $|\phi_{01}^{01}\rangle$ and $|\phi_{10}^{10}\rangle$ become (3) for the same transmittances of BSs 1 and 4 and phase shift $\xi_4 = \pi$. The optimized solution, giving the highest probability of successful truncation, is found for the 50:50 BSs ($t_1^2 = t_4^2 = 1/2$) in all four cases. It is worth noting that in [5, 6], the internal phases of both BSs are chosen as $\theta_t = 0$ and $\theta_r = \pi/2$, so to obtain the exact equivalence of the original scheme with ours, it is enough to set the phase shift $\xi_4 = \pi$.

As shown by Koniorczyk *et al* [8], the Pegg–Phillips–Barnett scissors device enables also the truncation of an arbitrary incident state to the qutrit state

$$|\phi_{\text{trun}}^{(3)}\rangle \sim \gamma_0 |0\rangle + \gamma_1 |1\rangle + \gamma_2 |2\rangle \qquad (23)$$

which can be obtained in our setup by assuming the single-photon Fock states in the input modes $\hat{a}_1$ and $\hat{a}_2$, together with the single-photon counts in detectors D_2 and D_4. Under these assumptions and by denoting $f_i' \equiv r_i^2 - t_i^2$, the output state in mode $\hat{b}_1$ becomes

$$|\phi_{11}^{11}\rangle \sim 2 r_1 t_1 r_4 t_4 (e^{2i\xi_4} \gamma_0 |0\rangle + \gamma_2 |2\rangle) + e^{i\xi_4} f_1' f_4' \gamma_1 |1\rangle \quad (24)$$

which is the desired truncated state if the parameters of BS1 and BS4 are related as follows:

$$t_4^2 = \frac{1}{2}\left(1 \pm \frac{r_1 t_1}{\sqrt{1 - 3(r_1 t_1)^2}}\right) \qquad (25)$$

and the phase shift ξ_4 is equal to π or zero, respectively. By inspection of (25) one readily finds that the optimum solutions occur for t_1^2 equal either to $(3 - \sqrt{3})/6 \approx 0.21$ or to $(3 + \sqrt{3})/6 \approx 0.79$ and $t_4^2 = t_1^2$ if $\xi_4 = 0$, in agreement with the results of [8], but also for $t_4^2 = 1 - t_1^2$ if $\xi_4 = \pi$.

4. Truncation to quartit states

In this section, we demonstrate how to realize truncation of an input state $|\psi\rangle$, given by (1), to the four-dimensional qudit

$$|\phi_{\text{trun}}^{(4)}\rangle \sim \gamma_0 |0\rangle + \gamma_1 |1\rangle + \gamma_2 |2\rangle + \gamma_3 |3\rangle \qquad (26)$$

referred to as the *quartit*. We apply the GQSD shown in figure 2, assuming that modes $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ are the in single-photon Fock states, and single photons have been measured in all detectors, $N_2 = N_3 = N_4 = 1$. If the light to be truncated enters the interferometer in mode $\hat{a}_4$, then the output state $|\phi\rangle = |\phi_{111}^{111}\rangle$ obtained via the projection synthesis is given by (17) for $d = 4$ and the amplitudes

$$c_n^{(4)} \equiv \langle n111|\hat{U}|111n\rangle \qquad (27)$$

depend on the BS and PS parameters. By applying the procedure described in section 2, we find that the simplest solution is for $n = d - 1$:

$$c_3^{(4)} = 6e^{2i\xi_2} r_1 t_1 r_2 t_2^2 r_4 t_4^2 r_5 t_5. \qquad (28)$$

It is seen that $c_3^{(4)}$ is independent of the BS3 parameters, so for simplicity let us assume that BS3 is removed ($t_3 = 1, \xi_3 = 0$). Thus, instead of a general setup, shown in figure 2, we first analyse its simplified version shown in figure 3(b). Under the assumption, the solutions for the other amplitudes are found to be

$$
\begin{aligned}
c_0^{(4)} &= -2e^{i(\xi_4 + \xi_5)} t_2 t_4 (e^{i(\xi_2 + \xi_5)} f_1' r_2 r_5 t_5 + e^{i\xi_4} f_5' r_1 t_1 r_4), \\
c_1^{(4)} &= -2 r_1 t_1 r_2 r_4 r_5 t_5 (e^{2i(\xi_2 + \xi_5)} f_2'' + e^{2i\xi_4} f_4'') \\
&\quad + e^{i(\xi_2 + \xi_4 + \xi_5)} f_1' f_2' f_4' f_5', \\
c_2^{(4)} &= 2e^{i\xi_2} t_2 t_4 (e^{i(\xi_2 + \xi_5)} r_1 t_1 g_2 r_4 f_5' + e^{i\xi_4} f_1' r_2 g_4 r_5 t_5)
\end{aligned}
\qquad (29)
$$

where, for brevity, the n-primed f denotes $r_k^2 - n t_k^2$, $g_k \equiv 2r_k^2 - t_k^2$, and the global phase factor $\exp(i\xi_1)$, the same for all $c_n^{(4)}$, was omitted as it cancels out during the renormalization with $\mathcal{N}$. Our multiport interferometer can act as a good quantum scissors device if there exist parameters $\mathbf{T}$ and $\boldsymbol{\xi}$ such that condition (20) is satisfied. As explained in section 2.3, we focus on the simplicity of the solutions, although we realize that they are not optimal as implied by the results of our numerical experiments. For example, a simple solution is found for the transmittances equal to

$$\mathbf{T} = \left[\tfrac{1}{3}, \tfrac{1}{4}, 1, \tfrac{1}{3}, \tfrac{1}{2}\right] \qquad (30)$$

and zero phase shifts $\boldsymbol{\xi}$ except $\xi_5 = \pi/2$. By applying these values to (29), one readily finds that $c_n^{(4)} = 1/12$ for $n = 0, 1, 2, 3$. Another solution of (20) is found for the transmittances

$$\mathbf{T} = \left[\tfrac{1}{26}(13 - 3\sqrt{13}), \tfrac{1}{2}, 1, \tfrac{1}{3}, \tfrac{1}{4}(2 + \sqrt{3})\right] \qquad (31)$$

and $\boldsymbol{\xi} = 0$, which results in a constant amplitude $c_n^{(4)} = 1/(4\sqrt{39})$, although this is much lower than that found for the first solution. Numerically it is easy to find solutions which correspond to higher $c_n^{(4)}$ than those given by our analytical solutions. For example, by choosing $T = [0.784\,94, 0.690\,01, 1, 0.874\,51, 0.701\,85]$ and zero

phase shifts $\boldsymbol{\xi}$ except $\xi_5 = \pi$, then the amplitudes $c_n^{(4)}$ for the output state $|\phi\rangle = |\phi_{111}^{111}\rangle$ are constant at the value of 0.134.

A closer look at the amplitudes (29) reveals that they remain unchanged by some permutations of transmittances accompanied by a proper change in the phase shifts, or by replacement of t_k^2 by $1 - t_k^2$. In particular, we find the following relations:

$$c_n^{(4)}(\mathbf{T}, \boldsymbol{\xi}) = c_n^{(4)}([t_5^2, t_4^2, 1, t_2^2, t_1^2], \boldsymbol{\xi}) \qquad (32)$$

if $\xi_1 = 0, \xi_4 = \xi_2 + \xi_5$, and

$$\begin{aligned} c_n^{(4)}(\mathbf{T}, \boldsymbol{\xi}) &= c_n^{(4)}([1 - t_1^2, t_2^2, 1, t_4^2, t_5^2], \boldsymbol{\xi}') \\ &= c_n^{(4)}([t_1^2, t_2^2, 1, t_4^2, 1 - t_5^2]\boldsymbol{\xi}'') \end{aligned} \qquad (33)$$

if $\boldsymbol{\xi} = \boldsymbol{\xi}' = \boldsymbol{\xi}''$ except for $\xi_2' = \xi_2 + \pi$ or, equivalently, $\xi_4' = \xi_4 + \pi$, and $\xi_5'' = \xi_5 + \pi$. Thus, by transforming solutions (30) and (31) according to (32) and (33) one can find new solutions.

5. Selective truncations

Here, we focus on generalized truncations of the input state $|\psi\rangle$ into a finite superposition of the form $|\phi_{\text{trun}}^{(d)}\rangle$, albeit with some states (say $|k_1\rangle, |k_2\rangle, \ldots$) removed, i.e.,

$$|\psi\rangle \to |\phi_{\text{holes}\,k_1,k_2,\ldots}^{(d)}\rangle = \mathcal{N} \sum_{\substack{n=0 \\ (n\neq k_1,k_2,\ldots)}}^{d-1} \gamma_n |n\rangle \qquad (34)$$

corresponding to the case when the amplitudes $c_n^{(d)}$ vanish for $n = k_1, k_2, \ldots$ and are constant but nonzero for the other n. This kind of quantum state engineering can be interpreted as truncation with hole burning. In general, *hole burning* in the Fock space of a given state of light, according to Baseia *et al* (see [44] and references therein) and Gerry and Benmoussa [45], means selective removal of one or more specific Fock states from the field. Although, originally, hole burning was applied to infinite-dimensional states, given by (1), this concept can also be used in the case of finite-dimensional states $|\phi_{\text{trun}}^{(d)}\rangle$. Alternatively, following the interpretation of D'Ariano *et al* [43], one can refer to the above quantum state engineering, especially when the number of holes exceeds $(d-1)/2$, as a kind of *Fock-state filtering*, which enables selection of some Fock states (say $|j_1\rangle, |j_2\rangle, \ldots$) from a given input state $|\psi\rangle$, i.e.,

$$|\psi\rangle \to |\phi_{\text{filter}\,j_1,j_2,\ldots}\rangle = \mathcal{N}(\gamma_{j_1}|j_1\rangle + \gamma_{j_2}|j_2\rangle + \cdots). \qquad (35)$$

Here, we show how the GQSD can be used for Fock-state filtering and hole burning in the case of $d = 4$. As the first example, we analyse truncation to $|\phi_{\text{trun}}^{(4)}\rangle$ with the two-photon Fock state removed which results in

$$|\phi_{\text{hole}\,2}^{(4)}\rangle \sim \gamma_0|0\rangle + \gamma_1|1\rangle + \gamma_3|3\rangle. \qquad (36)$$

We find that state $|\phi_{111}^{111}\rangle$, with amplitudes given by (29), is reduced to (36) for various transmittances $\mathbf{T}$ and phase shifts $\boldsymbol{\xi}$, including the following:

$$\mathbf{T} = \left[\tfrac{1}{14}(7 + \sqrt{21}), \tfrac{1}{3}, 1, \tfrac{1}{2}, \tfrac{1}{10}(5 - \sqrt{5})\right] \qquad (37)$$

and $\boldsymbol{\xi} = 0$. Similarly, the other truncated states with a single hole,

$$|\phi_{\text{hole}\,0}^{(4)}\rangle \sim \gamma_1|1\rangle + \gamma_2|2\rangle + \gamma_3|3\rangle, \qquad (38a)$$

$$|\phi_{\text{hole}\,1}^{(4)}\rangle \sim \gamma_0|0\rangle + \gamma_2|2\rangle + \gamma_3|3\rangle, \qquad (38b)$$

can be generated by the GQSD from $|\phi_{111}^{111}\rangle$ if, for example, $\boldsymbol{\xi} = 0$ and the transmittances are as follows:

$$\mathbf{T} = \left[\tfrac{1}{14}(7 + \sqrt{21}), \tfrac{1}{3}, 1, \tfrac{1}{2}, \tfrac{1}{4}(2 - \sqrt{2})\right], \qquad (39a)$$

$$\mathbf{T} = \left[\tfrac{1}{2} - \tfrac{3}{2}\sqrt{\tfrac{5}{173}}, \tfrac{1}{2}, 1, \tfrac{1}{6}, \tfrac{1}{2} + \tfrac{5}{2}\sqrt{\tfrac{3}{203}}\right], \qquad (39b)$$

respectively. One can check that superpositions of any two Fock states $|k\rangle$ and $|l\rangle$ for $k, l = 0, \ldots, 3$, i.e.,

$$|\phi_{\text{filter}\,kl}\rangle \sim \gamma_k|k\rangle + \gamma_l|l\rangle, \qquad (40)$$

can be obtained as special cases of $|\phi_{111}^{111}\rangle$, for example, for $\boldsymbol{\xi} = 0$, and the transmittances given by

$$|\phi_{\text{filter}\,02}\rangle : \mathbf{T} = [1, \tfrac{1}{2}, 1, 1, \tfrac{1}{2}],$$

$$|\phi_{\text{filter}\,03}\rangle : \mathbf{T} = \left[\tfrac{1}{2}\left(1 - \sqrt{\tfrac{5}{133}}\right), \tfrac{1}{2}, 1, \tfrac{1}{6}, \tfrac{1}{2} + \tfrac{3}{2}\sqrt{\tfrac{3}{155}}\right],$$

$$|\phi_{\text{filter}\,13}\rangle : \mathbf{T} = \left[\tfrac{1}{2}, \tfrac{1}{3}(3 - \sqrt{3}), 1, \tfrac{1}{3}(3 - \sqrt{3}), \tfrac{1}{2}\right], \qquad (41)$$

$$|\phi_{\text{filter}\,23}\rangle : \mathbf{T} = \left[\tfrac{1}{2}\left(1 - \sqrt{\tfrac{5}{37}}\right), \tfrac{1}{2}, 1, \tfrac{1}{6}, \tfrac{1}{2}\left(1 + \sqrt{\tfrac{3}{35}}\right)\right].$$

Note that our exemplary state $|\phi_{\text{filter}\,02}\rangle$, given in (41), can be realized in the Pegg–Phillips–Barnett scheme, since the chosen transmittances are a special case of (21). All the above examples were found for $t_3 = 1$. But we have not found solutions $|\phi_{\text{filter}\,12}\rangle$ by assuming $t_3 = 1$, together with the single-photon states in the input modes $\hat{a}_1, \hat{a}_2, \hat{a}_3$ and the single-photon counts in detectors D_2, D_3, and D_4. While keeping the latter two requirements, one can change only the transmittance of BS3. Then, we find a solution

$$|\phi_{\text{filter}\,12}\rangle : \mathbf{T} = \left[\tfrac{1}{2} + \tfrac{1}{\sqrt{5}}, \tfrac{8}{9}, \tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{\sqrt{5}}, 1\right] \qquad (42)$$

and $\boldsymbol{\xi} = 0$. On the other hand, the state $|\phi_{\text{filter}\,12}\rangle$ can be realized in the QSD with $t_3 = 1$, for example, as a special case of the output state $|\phi_{110}^{101}\rangle$ for

$$|\phi_{\text{filter}\,12}\rangle : \mathbf{T} = \left[\tfrac{1}{10}(5 - \sqrt{15}), \tfrac{2}{3}, 1, \tfrac{1}{2}, \tfrac{1}{2}\right] \qquad (43)$$

and $\boldsymbol{\xi} = 0$. The scheme also enables a synthesis of Fock states via teleportation. From $|\phi_{111}^{111}\rangle$, one can synthesize the two- and three-photon Fock states for the following transmittances:

$$|2\rangle : \mathbf{T} = \left[1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{2}, 1\right], \qquad (44a)$$

$$|3\rangle : \mathbf{T} = \left[\tfrac{1}{2}, \tfrac{1}{2}, 1, \tfrac{1}{2}, \tfrac{1}{2}\right] \qquad (44b)$$

and, for example, $\boldsymbol{\xi} = 0$ except $\xi_5 = \pi/2$ in the latter case. Actually, with the choice (44a), the state $|2\rangle$ is generated for arbitrary phase shifts. The two-photon Fock state cannot be obtained from $|\phi_{111}^{111}\rangle$ assuming $t_3 = 1$, which can be shown analytically. However, the state $|2\rangle$ can easily be obtained even for $t_3 = 1$ but from other states e.g. $|\phi_{110}^{101}\rangle$. It is worth noting that by applying the transformations given by (32) and (33) to the above solutions for $\mathbf{T}$, one can easily obtain new analytical solutions for the generation of states $|\phi_{\text{hole}\,k}^{(4)}\rangle$ and $|\phi_{\text{filter}\,kl}\rangle$. We

have given only some specific examples of $\mathbf{T}$, which guarantee the desired truncation. Although it is outside the main goal of this paper, it is possible to give more general conditions for $\mathbf{T}$, for example: (i) for any $\mathbf{T} = [t_1^2, t_2^2, 1, 1, 1/2]$ with $t_1^2 \neq 1/2$ and $t_2 \neq 0, 1$, the output state is $|\phi_{\text{filter}\,02}\rangle$, (ii) for any $\mathbf{T} = [1/2, t_2^2, 1, 1 - 1/(3r_2^2), 1/2]$ with $t_2 \in (0, 2/3)$ the output state is $|\phi_{\text{filter}\,13}\rangle$, assuming in both cases $\boldsymbol{\xi} = 0$. As already emphasized, the solutions presented here are usually not optimized, but they are simple enough to show analytically that the specific truncations can be realized by the GQSD.

6. Truncation to five-dimensional qudit states

A question arises whether the GQSD, shown in figure 3(b) with the removed BS3, can be used for truncation of the input state up to more than quartits. So, first we analyse possibilities of the truncation of an input state $|\psi\rangle$, given by (1), to the qudit state $|\phi_{\text{trun}}^{(5)}\rangle$ being a special case of (2) for $d = 5$. As usual, we assume that the light to be truncated is in mode $\hat{a}_4$, and the input modes $\hat{a}_1$ and $\hat{a}_3$ are in single-photon states, but, in contrast to the former sections, we choose mode $\hat{a}_2$ to be in the two-photon state. So, the total input state is $|\Psi\rangle = |121\psi\rangle$. We assume that the conditional measurement yields single-photon counts in detectors D_2 and D_4, but a two-photon count in D_3, thus the resulting output state $|\phi\rangle = |\phi_{121}^{121}\rangle$ is given by (18) for $d = 5$ and the amplitudes

$$c_n^{(5)} \equiv \langle n121|\hat{U}|121n\rangle \qquad (45)$$

equal to

$$
\begin{aligned}
c_0^{(5)} &= \mathrm{e}^{\mathrm{i}(\xi_4+\xi_5)} t_2 t_4 (3\mathrm{e}^{2\mathrm{i}(\xi_2+\xi_5)} f_1'' r_1 r_2^2 r_5 t_5^2 \\
&\quad + 2\mathrm{e}^{\mathrm{i}(\xi_2+\xi_4+\xi_5)} g_1 t_1 r_2 r_4 g_5 t_5 + 3\mathrm{e}^{2\mathrm{i}\xi_4} r_1 t_1^2 r_4^2 r_5 f_5''), \\
c_1^{(5)} &= 3 r_1 t_1 r_2 r_4 r_5 t_5 (\mathrm{e}^{3\mathrm{i}(\xi_2+\xi_5)} r_1 f_2'' r_2 t_5 + \mathrm{e}^{3\mathrm{i}\xi_4} t_1 f_4''' r_4 r_5) \\
&\quad - \mathrm{e}^{\mathrm{i}(\xi_2+\xi_4+\xi_5)} (\mathrm{e}^{\mathrm{i}(\xi_2+\xi_5)} f_1'' r_1 f_2'' r_2 f_4' g_5 t_5 \\
&\quad + \mathrm{e}^{\mathrm{i}\xi_4} g_1 t_1 f_2' f_4'' r_4 f_5'' r_5), \\
c_2^{(5)} &= \mathrm{e}^{\mathrm{i}\xi_2} t_2 t_4 \{2 t_1 r_2 r_4 t_5 [3\mathrm{e}^{2\mathrm{i}\xi_4} (r_1^2 t_4^2 - r_1^2 g_4 + t_1^2 f_4') r_5^2 \\
&\quad - \mathrm{e}^{2\mathrm{i}(\xi_2+\xi_5)} r_1^2 (f_2'' g_5 + g_2 f_5' + g_2 r_5^2)] \\
&\quad + \mathrm{e}^{\mathrm{i}(\xi_2+\xi_4+\xi_5)} r_1 r_5 [2 r_1^2 g_2 r_4^2 f_5' - 2 t_1^2 (f_2' + r_2^2) g_4 f_5'' \\
&\quad - r_1^2 g_2 (t_4^2 f_5'' + 2 r_4^2 t_5^2)]\}, \\
c_3^{(5)} &= 3\mathrm{e}^{2\mathrm{i}\xi_2} r_1 t_2^2 t_4^2 r_5 [\mathrm{e}^{\mathrm{i}(\xi_2+\xi_5)} r_1 t_1 (3r_2^2 - t_2^2) r_4 f_5'' \\
&\quad + \mathrm{e}^{\mathrm{i}\xi_4} f_1'' r_2 (3r_4^2 - t_4^2) r_5 t_5], \\
c_4^{(5)} &= 12\mathrm{e}^{3\mathrm{i}\xi_2} r_1^2 t_1 r_2^3 r_4 t_4^3 t_5 r_5^2.
\end{aligned}
\qquad (46)
$$

As in (29), the irrelevant global phase factor $\exp(\mathrm{i}\xi_1)$ was cancelled out from all $c_n^{(5)}$ in (46). We have not found analytical solutions for the BS and PS parameters satisfying condition (20) with $d = 5$ for the amplitudes given by (46), as the problem requires finding roots of sixth-order equations. Thus, we have applied a numerical procedure for finding the BS parameters for which $\Delta = 0$ with precision of the order of 10^{-16}. We have found various solutions including that for transmittances equal to $\mathbf{T} = [0.304\,64, 0.387\,75, 1, 0.817\,40, 0.184\,38]$, $\xi_4 = \pi$ and the

other phase shifts set to zero, which results in the constant amplitude $c_n^{(5)}$ for $n = 0, \ldots, 4$. By placing BS3 in the setup with $t_3 \neq 1$ one can find other solutions with larger constant $c_n^{(5)}$. However, since we are not interested in the optimalization but rather the simplicity of the scheme, we do not present these solutions here.

7. Truncation to six-dimensional qudit states

The eight-port QSD, shown in figure 2, enables truncation of the incident light $|\psi\rangle$ even to the six-dimensional qudit state $|\phi_{\text{trun}}^{(6)}\rangle$ as a special case of (2) for $d = 6$. To achieve perfect truncation, we assume a single-photon Fock state in mode $\hat{a}_2$, two-photon states in modes $\hat{a}_1, \hat{a}_3$, and that the conditional measurement yields $N_2 = N_4 = 2$ and $N_3 = 1$ photon counts. Then, the output state $|\phi\rangle = |\phi_{212}^{212}\rangle$ is given by (18) for $d = 6$ and the amplitudes

$$c_n^{(6)} \equiv \langle n212|\hat{U}|212n\rangle. \qquad (47)$$

The simplest-form amplitude is for $n = d - 1$, which reads as

$$c_5^{(6)} = 30\mathrm{e}^{2\mathrm{i}\xi_1+3\mathrm{i}\xi_2} r_1 t_1^2 r_2^3 t_2^2 r_4^2 t_4^3 r_5 t_5^2. \qquad (48)$$

As in the former cases for the truncation to qudits with $d < 6$, the amplitude $c_{d-1}^{(d)}$ is independent of the BS3 parameters. Unfortunately, contrary to the former cases, we have not found numerically any solutions satisfying $c_n^{(6)} = \text{const}$ for $n = 0, \ldots, d - 1$ for the simplified scheme with removed BS3, shown in figure 3(b). Thus, we analyse again the general scheme shown in figure 2 with $t_3 \neq 0, 1$. As an example, we give a solution for $c_4^{(6)}$ to be as follows:

$$
\begin{aligned}
c_4^{(6)} &= 6\mathrm{e}^{2\mathrm{i}(\xi_1+\xi_2)} t_1 r_2 t_2^2 r_4 t_4^2 t_5 \{(3r_4^2 - 2t_4^2) \\
&\quad \times \mathrm{e}^{\mathrm{i}\xi_4} [\mathrm{e}^{\mathrm{i}\xi_2} r_1 t_1 (3r_2^2 - 2t_2^2) r_3 + \mathrm{e}^{\mathrm{i}\xi_3} g_1 r_2 t_3] r_5 t_5 \\
&\quad + \mathrm{e}^{\mathrm{i}\xi_5} [\mathrm{e}^{\mathrm{i}\xi_2} r_1 t_1 (3r_2^2 - 2t_2^2) t_3 - \mathrm{e}^{\mathrm{i}\xi_3} g_1 r_2 r_3] r_4 g_5\}. \qquad (49)
\end{aligned}
$$

Solutions for $c_n^{(6)}$ with $n = 0, \ldots, 3$ are quite lengthy, so we do not present them explicitly here. Nevertheless, they have been used in our numerical search of the BS and PS parameters satisfying condition (20) for $d = 6$. Thus, we have found various solutions for which $\Delta \sim 10^{-16}$. We just mention a solution for the BS transmittances equal to $\mathbf{T} = [0.755\,72, 0.417\,83, 0.325\,03, 0.832\,74, 0.503\,38]$, $\xi_4 = \pi$ and the other phase shifts equal to zero, which results in the constant nonzero amplitude $c_n^{(6)}$ for $n = 0, \ldots, 5$, and is obviously vanishing for $n > 5$. Another solution is found for the same phase shifts but transmittances equal to $\mathbf{T} = [0.581\,54, 0.285\,19, 0.467\,53, 0.685\,58, 0.498\,36]$, which also results in a constant $c_n^{(6)}$ but one that is slightly lower than that in the former case.

8. Imperfect photon counting

Now, we address an important problem from an experimental point of view that concerns the deterioration effects of system imperfections on the fidelity of truncation and teleportation. To analyse such effects one can follow the approaches of, for example, [6, 7, 12–14, 50] applied to the Pegg–Phillips–Barnett QSD. Here, we focus on imperfect photodetection.

Photon counting in mode $\hat{b}_i$ by an imperfect detector with a finite efficiency η_i and a mean dark count rate ν_i can be described by a positive-operator-valued measure (POVM) [51] with the following elements [50]:

$$\hat{\Pi}_{N_i}^{(b_i)} = \sum_{n=0}^{N_i} \sum_{m=n}^{\infty} \frac{e^{-\nu_i} \nu_i^{N_i-n}}{(N_i-n)!} \eta_i^n (1-\eta_i)^{m-n} C_n^m |m\rangle_i{}_i\langle m| \quad (50)$$

summing up to the identity operator $\hat{I}$. In (50), N_i is the number of registered photocounts in detector D_i, n is the actual number of photons entering the detector, $N_i - n$ is the number of dark counts, and C_n^m are binomial coefficients. The mean dark count rate ν in (50) is related to the standard dark count rate R_{dark} by the relation $\nu = \tau_{\text{res}} R_{\text{dark}}$, where τ_{res} is the detector resolution time. In analogy to the analysis of the Pegg–Phillips–Barnett QSD given in [12], we compare the fidelities for the states truncated by the generalized scissors in relation to three types of applied detectors. (i) Conventional photodetectors (e.g., avalanche photo-diodes, APDs) providing only a binary answer to the question whether any photons have been registered or not, thus described by a POVM with the two elements:

$$\hat{\Pi}_{c0}^{(b_i)} = \hat{\Pi}_0^{(b_i)}, \qquad \hat{\Pi}_{c1}^{(b_i)} = \hat{I} - \hat{\Pi}_0^{(b_i)}. \quad (51)$$

(ii) Single-photon resolving photodetectors (e.g., visual light photon counters, VLPCs [52]) providing a trinary answer to the question whether zero, one or more than one photons have been registered, so given by a POVM with the following three elements:

$$\hat{\Pi}_{s0}^{(b_i)} = \hat{\Pi}_0^{(b_i)}, \qquad \hat{\Pi}_{s1}^{(b_i)} = \hat{\Pi}_1^{(b_i)},$$
$$\hat{\Pi}_{s2}^{(b_i)} = \hat{I} - \hat{\Pi}_0^{(b_i)} - \hat{\Pi}_1^{(b_i)} \quad (52)$$

where $\hat{\Pi}_{0,1}^{(b_i)}$ in (51) and (52) are given by (50). (iii) Unrealistic detectors (labelled by r) resolving any number of simultaneously absorbed photons described by the POVM elements $\hat{\Pi}_{rN_i}^{(b_i)} \equiv \hat{\Pi}_{N_i}^{(b_i)}$, given by (50) for any N_i. Such detectors are not available, although some methods (including the so-called photon chopping [53]) have been proposed to measure photon statistics with conventional devices. By including the imperfect photon counting by detectors D_2, D_3 and D_4, the output state at mode $\hat{b}_1$ can be described by the following density matrix:

$$\hat{\rho}_x = \mathcal{N} \text{Tr}_{(b_2,b_3,b_4)} \left(\hat{\Pi}_{xN_2}^{(b_2)} \hat{\Pi}_{xN_3}^{(b_3)} \hat{\Pi}_{xN_4}^{(b_4)} |\Phi\rangle\langle\Phi| \right), \quad (53)$$

where the partial trace is taken over the detected modes $\hat{b}_2$, $\hat{b}_3$, and $\hat{b}_4$; $\hat{\Pi}_{xN_i}^{(b_i)}$ are the POVM elements for a given type of detector $x = c, s, r$; $|\Phi\rangle$ is the four-mode output state, given by (12), and $\mathcal{N}$ is the normalization. For simplicity, we can assume identical detectors with $\eta \equiv \eta_2 = \eta_3 = \eta_4$ and $\nu \equiv \nu_2 = \nu_3 = \nu_4$. Deviation of the realistically truncated state $\hat{\rho}_x$ from the ideally truncated state $|\phi\rangle$ is usually described by the fidelity

$$F_x = \langle\phi|\hat{\rho}_x|\phi\rangle. \quad (54)$$

In our numerical analysis we assume:

(i) $\eta = 0.7$ and $R_{\text{dark}} \sim 100 \text{ s}^{-1}$ for conventional detectors (see e.g. [12]),

(ii) $\eta = 0.88$ and $R_{\text{dark}} = 10^4 \text{ s}^{-1}$ for single-photon detectors (VLPCs) as experimentally achieved by Takeuchi *et al* [52],

(iii) for theoretic photon-number resolving detectors we choose the same η and R_{dark} as in (ii).

Moreover, we put $\tau_{\text{res}} \sim 10$ ns. We observe that the truncation fidelity in the system with imperfect photodetection depends on the chosen transmittances. In particular, the different solutions described in section 4 for perfect truncation (with $F = 1$) in the lossless system correspond not only to different probabilities of success but also to different fidelities of truncation in the lossy system. For $\alpha = 0.4$, we find that the fidelity for truncation up to quartit states in the system described by the transmittances given below equation (31) drop from one to $F_c \approx 0.91$ for the conventional detectors and to $F_s = F_r \approx 0.98$ for the VLPCs and the photon-number resolving detectors. The fidelities of truncation up to five-dimensional states in the system described in section 6 are estimated to be $F_c \approx 0.67$ for the conventional detectors, $F_s \approx 0.95$ for the VLPCs, and $F_r \approx 0.96$ for the photon-number resolving detectors if $\alpha = 0.4$. These estimations show that conventional photodetectors can effectively be used for the low-intensity-field truncations described in sections 3–5, where at most single-photon detections are required. In the schemes described in sections 6 and 7, where detections of two-photons are important, one has to apply at least single-photon resolving detectors even in the low-intensity limit. It is worth noting that by increasing the number of detectors and beam splitters in the discussed pyramid configuration, one can achieve truncations to higher-dimensional states by detecting no or single photons only [54]. In our estimation we have assumed, based on [52], relatively high values of the dark count rates. However, it has recently been experimentally demonstrated by Babichev *et al*, by rigorously synchronizing the photon count events, that the dark counts can be reduced to a negligible level [15]. As multiport optical interferometers have already been experimentally realized [40–42], it seems that the proposed GQSD for the truncation and teleportation of at least quartit states is accessible to experiments with present-day technology for low-intensity incident fields.

9. Discussion and conclusions

We are aware that our analysis of quantum state truncation via a GQSD is not yet complete. Among open problems to be analysed in greater detail we should mention:

(i) A generalization of the scheme for the truncation of an arbitrary d-dimensional qudit state based on the $2N$-port interferometer in the triangle configuration with the top BS removed. By symmetry of the scheme, shown in figure 2, such a generalization is straightforward, for example, along the lines of [35]. However, it would be desirable to find the minimum number of detections in the multiport QSD, which enables the state truncation to a qudit state of a given dimensionality.

(ii) An analysis of other kinds of losses (including mode mismatch in addition to imperfect photon-counting) in the generalized optical state truncation.

(iii) A detailed experimental proposal of the scheme for truncation at least to qutrit and quartit states.

(iv) A hard problem is the optimalization of solutions for the BS parameters to obtain the highest probability of truncation for $d \geqslant 4$.

These problems are currently under our investigation [54].

In conclusion, we have proposed a generalization of the Pegg–Phillips–Barnett six-port QSD (shown in figure 1) to the eight-port optical interferometer, depicted in figure 2. The analysed system enables, upon post-selection based on photon counting results, generation and teleportation of qudit states (for $d = 2, \dots, 6$) by truncation of an input optical field at the $(d-1)$th term of its Fock-state expansion. We have discussed examples of selective truncations, which enable Fock-state filtering and hole burning in the Fock space of an input optical field. We have also analysed the deterioration of the truncation fidelity due to realistic photon counting including finite photon-number resolution, inefficiency and dark counts of photodetectors. Our estimations suggest that the scheme is experimentally feasible at least for the generation and teleportation of quartit states of low-intensity incident fields.

Acknowledgments

References

[1] 1997 Quantum state preparation and measurement *J. Mod. Opt.* **44** (11/12) (Special issue)

[2] Knill E, Laflamme R and Milburn G J 2001 *Nature* **409** 46

[3] O'Brien J L, Pryde G J, White A G, Ralph T C and Branning D A 2003 *Nature* **426** 264

[4] Yamamoto T, Koashi M, Özdemir S K and Imoto N 2003 *Nature* **421** 343

[5] Pegg D T, Phillips L S and Barnett S M 1998 *Phys. Rev. Lett.* **81** 1604

[6] Barnett S M and Pegg D T 1999 *Phys. Rev.* A **60** 4965

[7] Villas-Bôas C J, de Almeida N G and Moussa M H Y 1999 *Phys. Rev.* A **60** 2759

[8] Koniorczyk M, Kurucz Z, Gabris A and Janszky J 2000 *Phys. Rev.* A **62** 013802

[9] Paris M G A 2000 *Phys. Rev.* A **62** 033813

[10] Guimaraes Y, Baseia B, Villas-Bôas C J and Moussa M H Y 2000 *Phys. Lett.* A **268** 260

[11] Villas-Bôas C J, Guimarăes Y, Moussa M H Y and Baseia B 2001 *Phys. Rev.* A **63** 055801

[12] Özdemir Ş K, Miranowicz A, Koashi M and Imoto N 2001 *Phys. Rev.* A **64** 063818
Özdemir Ş K, Miranowicz A, Koashi M and Imoto N 2002 *J. Mod. Opt.* **49** 977

[13] Özdemir Ş K, Miranowicz A, Koashi M and Imoto N 2002 *Phys. Rev.* A **66** 053809

[14] Miranowicz A and Leoński W 2004 *J. Opt. B: Quantum Semiclass. Opt.* **6** S43

[15] Babichev S A, Ries J and Lvovsky A I 2003 *Europhys. Lett.* **64** 1

[16] Resch K J, Lundeen J S and Steinberg A M 2002 *Phys. Rev. Lett.* **88** 113601

[17] Barnett S M and Pegg D T 1996 *Phys. Rev. Lett.* **76** 4148

[18] Phillips L S, Barnett S M and Pegg D T 1998 *Phys. Rev.* A **58** 3259

[19] Baseia B, Moussa M H Y and Bagnato V S 1997 *Phys. Lett.* A **231** 331

[20] Serra R M, de Almeida N G, Villas-Bôas C J and Moussa M H Y 2000 *Phys. Rev.* A **62** 043810

[21] Pregnell K L and Pegg D T 2002 *Phys. Rev.* A **66** 013810

[22] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895

[23] Miranowicz A and Tamaki K 2002 *Math. Sci. (Suri-Kagaku)* **473** 28

[24] Lee H W and Kim J 2000 *Phys. Rev.* A **63** 012305

[25] Bužek V, Wilson-Gordon A D, Knight P L and Lai W K 1992 *Phys. Rev.* A **45** 8079

[26] Miranowicz A, Piątek K and Tanaś R 1994 *Phys. Rev.* A **50** 3423

[27] Miranowicz A, Leoński W and Imoto N 2001 *Adv. Chem. Phys.* **119(I)** 155

[28] Leoński W and Tanaś R 1994 *Phys. Rev.* A **49** R20

[29] Leoński W 1997 *Phys. Rev.* A **55** 3874

[30] Miranowicz A, Leoński W, Dyrting S and Tanaś R 1996 *Acta Phys. Slov.* **46** 451

[31] Leoński W and Miranowicz A 2001 *Adv. Chem. Phys.* **119(I)** 195

[32] Leoński W and Miranowicz A 2004 *J. Opt. B: Quantum Semiclass. Opt.* **6** S37

[33] Zeilinger A, Bernstein H J, Greenberger D M, Horne M A and Żukowski M 1993 *Quantum Control and Measurement* ed H Ezawa and Y Murayama (Amsterdam: North-Holland) pp 9–22

[34] Zeilinger A, Żukowski M, Horne M A, Bernstein H J and Greenberger D M 1994 *Quantum Interferometry* ed F DeMartini and A Zeilinger (Singapore: World Scientific)

[35] Reck M, Zeilinger A, Bernstein H J and Bertani P 1994 *Phys. Rev. Lett.* **73** 58

[36] Jex I, Stenholm S and Zeilinger A 1995 *Opt. Commun.* **117** 95

[37] Żukowski M, Zeilinger A and Horne M A 1997 *Phys. Rev.* A **55** 2464

[38] Campos R A 2000 *Phys. Rev.* A **62** 013809

[39] Żukowski M and Kaszlikowski D 1999 *Phys. Rev.* A **59** 3200

[40] Mattle C, Michler M, Weinfurter H, Zeilinger A and Żukowski M 1995 *Appl. Phys.* B **60** S111

[41] Reck M 1996 *PhD Thesis* University of Innsbruck, unpublished

[42] Mohseni M, Steinberg A M and Bergou J A 2004 *Phys. Rev. Lett.* **93** 200403

[43] D'Ariano G M, Maccone L, Paris M G A and Sacchi M F 2000 *Phys. Rev.* A **61** 053817

[44] Escher B M, Avelar A T, da Rocha Filho T M and Baseia B 2004 *Phys. Rev.* A **70** 025801

[45] Gerry C C and Benmoussa A 2002 *Phys. Lett.* A **303** 30

[46] Campos R A, Saleh B E A and Teich M C 1989 *Phys. Rev.* A **40** 1371

[47] Leonhardt U 1997 *Measuring the Quantum State of Light* (Cambridge: Cambridge University Press) chapter 4

[48] Ralph T C, White A G, Munro W J and Milburn G J 2002 *Phys. Rev.* A **65** 012314

[49] Skaar J, García Escartín J C and Landro H 2004 *Am. J. Phys.* **72** 1385

[50] Barnett S M, Phillips L S and Pegg D T 1998 *Opt. Commun.* **158** 45

[51] Peres A 1993 *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer)

[52] Takeuchi S, Kim J, Yamamoto Y and Hogue H H 1999 *Appl. Phys. Lett.* **74** 1063

[53] Paul H, Törmä P, Kiss T and Jex I 1996 *Phys. Rev. Lett.* **76** 2464

[54] Miranowicz A, Özdemir Ş K, Bajer J, Koashi M and Imoto N 2005 in preparation