Comment on "Inseparability Criteria for Continuous Bipartite Quantum States"

In a recent Letter [1], Shchukin and Vogel (SV) derived the necessary and sufficient conditions for a bipartite continuous variable (CV) quantum state $\hat{\rho}$ to be positive under partial transposition (PPT). The SV result shows a common basis of other well-known CV inseparability criteria, some of which seemed previously to be independent of partial transposition. The argument used in Ref. [1] relates the condition of PPT of a bipartite state to the positivity of a corresponding (infinite) matrix of moments [see Eq. (12) of Ref. [1]]. From the latter, SV derived an infinite series of inequalities [Eq. (20) of Ref. [1]] and claimed that such a series of inequalities provides a necessary and sufficient condition for a state to be PPT (SV criterion). The latter claim is not correct, since inequalities (20) are necessary but not sufficient, as we prove by providing some counterexamples. We propose an amended version of the criterion.

The SV conditions were formulated in terms of a hierarchy of inequalities for the moments of creation and annihilation operators $M_{ij}(\hat{\rho}) \equiv M_{i_1i_2i_3i_4,j_1j_2j_3j_4}(\hat{\rho}) = \mathrm{Tr}[(\hat{a}^{\dagger i_1}\hat{a}^{i_2}\hat{b}^{\dagger i_3}\hat{b}^{i_4})^\dagger(\hat{a}^{\dagger j_1}\hat{a}^{j_2}\hat{b}^{\dagger j_3}\hat{b}^{j_4})\hat{\rho}]$. Here i stands for the multiindex (i_1, i_2, i_3, i_4) (analogously for j), so that we associate to i an operator given by $\hat{a}^{\dagger i_1}\hat{a}^{i_2}\hat{b}^{\dagger i_3}\hat{b}^{i_4}$. Moments form an infinite Hermitian matrix $M(\hat{\rho}) = [M_{ij}(\hat{\rho})]$. SV observed that the moments calculated for the partially transposed state $\hat{\rho}^\Gamma$ (say, with respect to the second subsystem) correspond to moments for $\hat{\rho}$ with reordered indices, i.e., $M_{i_1i_2i_3i_4,j_1j_2j_3j_4}(\hat{\rho}^\Gamma) = M_{i_1i_2j_3j_4,j_1j_2i_3i_4}(\hat{\rho})$. In Ref. [1], Eq. (12), it was found that $\hat{\rho}$ is PPT if and only if the (infinite) matrix $M(\hat{\rho}^\Gamma)$ is positive semidefinite.

Let us recall Sylvester's criterion (see, e.g., [2]). For any (possibly infinite) Hermitian matrix $\mathcal{M}$, let $\mathcal{M}^{\mathbf{r}}$, $\mathbf{r} = (r_1, \ldots, r_N)$ denote the submatrix which is obtained by deleting all rows and columns except the ones labeled by $r_1, \ldots, r_N$. Moreover, let $\mathcal{M}_N \equiv \mathcal{M}^{(1,2,\ldots,N)}$, i.e., the submatrix corresponding to the first N rows and columns. Then Sylvester's criterion can be formulated as follows: (i) $\mathcal{M}$ is positive definite if and only if all its *leading principal* minors are positive, i.e., $\det\mathcal{M}_N > 0$ for $N = 1, 2, \ldots$, while (ii) $\mathcal{M}$ is positive semidefinite if and only if all its *principal* minors are non-negative, i.e., $\det\mathcal{M}^{\mathbf{r}} \geq 0$ for any $\mathbf{r} \equiv (r_1, \ldots, r_N)$, $1 \leq r_1 < r_2 < \ldots < r_N$, and $N = 1, 2, \ldots$. In Ref. [1], Sylvester's criterion (i) was used, leading the authors to formulate incorrectly the following equivalent conditions:

$$\hat{\rho} \text{ is PPT} \Leftrightarrow \forall\ N\colon\ \det M_N(\hat{\rho}^\Gamma) \geq 0,$$

$$\hat{\rho} \text{ is NPT} \Leftrightarrow \exists\ N\colon\ \det M_N(\hat{\rho}^\Gamma) < 0,$$

given by Eqs. (20) and (21), where NPT stands for nonpositive under partial transposition. The SV entanglement criterion should be based on Sylvester's criterion (ii) rather than (i), since we deal with the request of positive semidefiniteness, which implies that SV conditions should be modified to include all subdeterminants of $M(\hat{\rho}^\Gamma)$ as:

$$\hat{\rho} \text{ is PPT} \Leftrightarrow \forall\ \mathbf{r}\colon\ \det M^{\mathbf{r}}(\hat{\rho}^\Gamma) \geq 0,$$

$$\hat{\rho} \text{ is NPT} \Leftrightarrow \exists\ \mathbf{r}\colon\ \det M^{\mathbf{r}}(\hat{\rho}^\Gamma) < 0.$$

One can show that the original SV criterion does not reveal that some exemplary states are NPT. Let us apply the SV criterion to the singlet state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, e.g., in a Fock basis. By using the same ordering of moments as in Ref. [1], we find that the determinant $\det M_N(\hat{\rho}^\Gamma)$ is strictly greater than zero for $N = 1, \ldots, 7$, and it vanishes for $N \geq 8$. Thus, according to the criterion as stated originally in Ref. [1], one could draw the conclusion that the Bell state is PPT. On the other hand, the inseparability of the Bell state is revealed by the modified criterion by choosing indices $\mathbf{r}$ corresponding to operators 1, $\hat{a}\hat{b}$. The same problem arises for other Bell states, for higher-dimensional states, and also for CV infinite-dimensional states. For example, one can define a CV Bell state as a superposition of coherent states $|\psi\rangle \propto |\alpha, \beta\rangle - |-\alpha, -\beta\rangle$. We find $\det M_N(\hat{\rho}^\Gamma)$ to be non-negative for $N = 1, \ldots, 7$ and vanishing for $N \geq 8$. By contrast, the entanglement of the state is revealed by selecting indices $\mathbf{r}$ corresponding to operators 1, $\hat{b}$, $\hat{a}\hat{b}$. In Ref. [1], the authors remark that it is possible to focus, for convenience (e.g., to involve a lower number of moments), just on some principal—and not necessarily leading—minors of $M(\hat{\rho}^\Gamma)$. Indeed, the authors detect the entanglement of the above mentioned CV Bell state exactly with the choice of indices we listed. In the same way, they rederived the other already mentioned criteria of entanglement in CV systems.

We stress that looking at principal minors—and not solely at the leading principal minors—corresponds exactly to the spirit of the amended criterion. It is clear that it is not only a matter of convenience, but it is necessary to take into account the case of singular matrices.

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