

Optimal mirror phase-covariant cloning

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We propose a quantum cloning machine, which clones a qubit into two clones assuming known modulus of expectation value of Pauli σ_z matrix. The process is referred to as the mirror phase-covariant cloning for which the input state is *a priori* less known than that for the standard phase-covariant cloning. Analytical expressions describing the cloning transformation and fidelity of the clones are found. Extremal equations for the optimal cloning are derived and analytically solved by generalizing a method of Fiurášek [Phys. Rev. A **64**, 062310 (2001)]. Quantum circuits implementing the optimal cloning transformation and their physical realization in a quantum-dot system are described.

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I. INTRODUCTION

One of the fundamental no-go theorems in quantum mechanics is the no-cloning theorem [1], which states that unknown quantum states cannot be perfectly copied. In other words, no quantum mechanical evolution exists which would transform a quantum state according to $|\phi\rangle \rightarrow |\phi\rangle|\phi\rangle$ for an unknown state $|\phi\rangle$. This theorem is a consequence of the linearity of quantum mechanics and it has tremendous technological implications; e.g., it is at the basis of the security of quantum communication protocols including quantum key distribution. Although exact cloning is impossible, pretty good approximate cloning is possible as shown for the first time by Bužek and Hillery [2]. They designed a cloning machine, referred to as the $1 \rightarrow 2$ universal cloner (UC), which produces two approximate copies from an unknown pure qubit state. The UC is a state-independent symmetric cloner in the sense that all qubit states are cloned with the same fidelity $F=5/6$ and fidelities of the clones to the initial pure state is the same $F_1=F_2$. The concept of cloning has attracted considerable interest and it has been later shown that for the $1 \rightarrow M$ UC, the relation between the optimum fidelity F of each copy and the number M of copies is given by $F=(2M+1)/(3M)$ [3]. Setting $M \rightarrow \infty$ corresponds to a classical cloning machine with $F=2/3$, which is the best fidelity that one can achieve with only classical operations. Moreover, the concept has been extended to include cloning of qudits, cloning of continuous-variable systems or state-dependent cloning (nonuniversal cloning), which can produce clones of a specific set of qubits with much higher fidelity than the rest [4–12] (for reviews see [13]). This paper is devoted to the latter topic.

Suppose we want to clone a qubit, which is in a pure state

$$|\psi\rangle = \cos\frac{\vartheta}{2}|0\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|1\rangle \quad (1)$$

parametrized by polar ϑ and azimuthal ϕ angles on the Bloch sphere. By considering only the $1 \rightarrow 2$ cloning, the joint density matrix of both clones can be given by [7,8]

$$\rho_{\text{out}} = \text{Tr}_{\text{in}}(\chi \rho_{\text{in}}^T \otimes \mathbb{1}_{\text{out}}), \quad (2)$$

where χ is a trace-preserving completely positive (TPCP) map describing the cloning operation in the tensor product of the input (\mathcal{H}_{in}) and output (\mathcal{H}_{out}) Hilbert spaces. Moreover, Tr_{in} stands for partial trace over \mathcal{H}_{in} , $\rho_{\text{in}} = |\psi\rangle\langle\psi|$, T denotes transposition, and $\mathbb{1}_{\text{out}}$ is the identity operator in \mathcal{H}_{out} . The quality of the cloning can be described by the single-clone fidelity

$$F_j(\vartheta, \phi) = \langle\psi|\rho_j|\psi\rangle, \quad (3)$$

where $\rho_j = \text{Tr}_{j \oplus 1}(\rho_{\text{out}})$ is the reduced density matrix of the j th clone ($j=1, 2$). As shown in [7], the map χ can be found by using an optimization procedure which maximizes the fidelity of the clones. In order to find a map χ , which makes clones of pure state qubits of the best possible quality using a partial knowledge about the states, one needs to maximize the average single-copy fidelity

$$F = \frac{1}{2} \int_0^{2\pi} d\phi \int_0^\pi d\vartheta g(\vartheta, \phi) [F_1(\vartheta, \phi) + F_2(\vartheta, \phi)], \quad (4)$$

which is an average over all possible input qubits defined by the distribution function $g(\vartheta, \phi)$.

The study of state-dependent cloning machines is important as it is often the case that we have some *a priori* information on the quantum state but we do not know it exactly. Using the available *a priori* information, we can then design a cloning machine which performs better cloning than the UC for some specific set of qubits. For example, if it is known that the qubit is chosen from the equator of the Bloch sphere, then we know that ϕ can be arbitrary while $\vartheta = \pi/2$. For such a case, the so-called *phase-covariant cloners* (PCCs) have been designed [6,9] and they have been shown to be optimal providing a higher fidelity than the UC. Fiurášek [7,8] studied the PCCs with known $\vartheta = \theta$ from the full range $[0, \pi]$ and provided two optimal symmetric cloners; one for the states in the lower and the other for those in the upper hemisphere of the Bloch sphere.

In this paper, we assume less *a priori* information and construct an optimal $1 \rightarrow 2$ symmetric cloner using the ap-

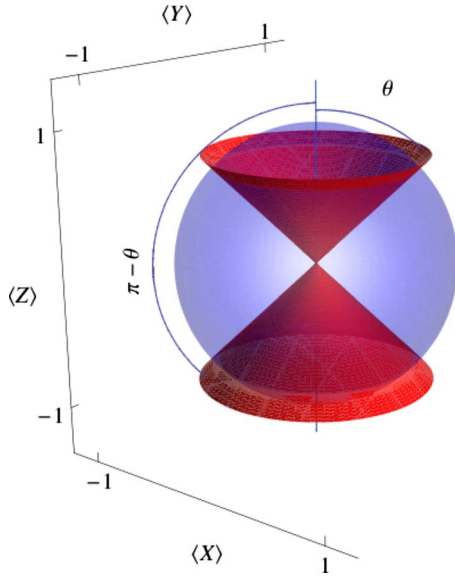


FIG. 1. (Color online) The intersection of two cones and a Bloch sphere provides the set of qubits that we want to clone in an optimal way. Here, X, Y, Z denote Pauli operators.

proach developed by Fiurášek [7,8]. Contrary to the works of Fiurášek, where the cloners are designed for known fixed value of θ , that is fixed σ_z component, we provide an optimal cloner for the qubits with known $\sin \theta$ (or, equivalently, $|\langle \sigma_z \rangle|$). Thus, our cloner prepares two symmetric clones for any qubit with $\vartheta = \theta$ or $\pi - \theta$.

II. OPTIMAL CLONING AND PARTIAL KNOWLEDGE ABOUT INITIAL STATE

Suppose that we are given a qubit prepared in a pure state $|\psi\rangle$, given by Eq. (1), together with the expectation value of the observable σ_z , that is ϕ is arbitrary but ϑ is equal either to θ or to $\pi - \theta$. Our task is to find a $1 \rightarrow 2$ cloning machine which prepares optimal approximate clones of the input qubit. The input qubits of interest are those lying along the intersection of two cones, which have the cone angles θ and $\pi - \theta$, sharing the same apex with Bloch sphere as shown in Fig. 1. Thus, the qubits are from both the upper and lower hemispheres of the Bloch sphere. We can consider the qubits from the lower hemisphere as the mirror images of those from the upper hemisphere or vice versa. Therefore, we suggest to call this cloning machine as the *mirror phase-covariant cloner* (MPCC) and require that it satisfies the following conditions: (i) qubits in the upper and lower hemisphere are cloned with the same maximal fidelity, $F(\theta) = F(\pi - \theta)$, and (ii) the sum of the fidelities of the two clones is the maximum attainable fidelity.

To derive the transformation we maximize the following functional:

$$F = \text{Tr}(\chi R), \quad (5)$$

where χ is a positive map and $R = \frac{1}{2}(r_\theta + r_{\pi-\theta})$ given in terms of

$$r_\theta = \frac{1}{8} \begin{pmatrix} 8c_2^4 & 0 & 0 & 0 & 0 & s_1^2 & s_1^2 & 0 \\ 0 & 4c_2^2 & 0 & 0 & 0 & 0 & 0 & s_1^2 \\ 0 & 0 & 4c_2^2 & 0 & 0 & 0 & 0 & s_1^2 \\ 0 & 0 & 0 & 2s_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2s_1^2 & 0 & 0 & 0 \\ s_1^2 & 0 & 0 & 0 & 0 & 4s_2^2 & 0 & 0 \\ s_1^2 & 0 & 0 & 0 & 0 & 0 & 4s_2^2 & 0 \\ 0 & s_1^2 & s_1^2 & 0 & 0 & 0 & 0 & 8s_2^4 \end{pmatrix}, \quad (6)$$

where $s_i = \sin(\theta/i)$ and $c_i = \cos(\theta/i)$ for $i = 1, 2$. The derivation of Eq. (5) is based on the method described in Ref. [8]. For the MPCC, g distribution occurring in Eq. (4) is given by

$$g_\theta(\vartheta, \phi) = \frac{1}{4\pi} [\delta(\vartheta - \theta) + \delta(\vartheta + \theta - \pi)], \quad (7)$$

in terms of Dirac's δ function. Moreover, subscript θ was added to indicate *a priori* knowledge about the input state. Note that $g_\theta(\vartheta, \phi)$ is equal to $\frac{1}{4\pi} \sin \vartheta$ for the UC and to $\frac{1}{2\pi} \delta(\vartheta - \theta)$ for the PCC. By defining a ϕ -averaged fidelity for each of the clones ($i = 1, 2$)

$$F_i(\theta) = \frac{1}{2\pi} \int_0^{2\pi} F_i(\theta, \phi) d\phi, \quad (8)$$

one can rewrite Eq. (4) assuming Eq. (7) as

$$F(\theta) = \frac{1}{4} [F_1(\theta) + F_2(\theta) + F_1(\pi - \theta) + F_2(\pi - \theta)], \quad (9)$$

and this quantity is to be maximized. By contrast, $F(\theta) = \frac{1}{2} [F_1(\theta) + F_2(\theta)]$ was applied in the maximization procedure for the PCC. Using the method given in [7] we derive map χ in the following form:

$$\chi(\theta) = \begin{pmatrix} A & 0 & 0 & 0 & 0 & C & C & 0 \\ 0 & B & B & 0 & 0 & 0 & 0 & C \\ 0 & B & B & 0 & 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 0 & B & B & 0 \\ C & 0 & 0 & 0 & 0 & B & B & 0 \\ 0 & C & C & 0 & 0 & 0 & 0 & A \end{pmatrix}, \quad (10)$$

where A, B , and C are θ dependent. Moreover, $B = (1 - A)/2$ as required by trace preservation of the transformation and $C = \sqrt{AB}$ as will be shown analytically in the following.

III. ANALYTICAL RESULTS

Using the Kraus decomposition with χ , as described in [7], we find that one of the possible unitary implementations of the TPCP map can be written as

$$|0\rangle|0\rangle_{\text{anc}} \rightarrow \frac{A|00\rangle|0\rangle_{\text{anc}} + \sqrt{2}C|\psi_+\rangle|1\rangle_{\text{anc}}}{\sqrt{A^2 + 2C^2}},$$

$$|1\rangle|0\rangle_{\text{anc}} \rightarrow \frac{A|11\rangle|1\rangle_{\text{anc}} + \sqrt{2}C|\psi_+\rangle|0\rangle_{\text{anc}}}{\sqrt{A^2 + 2C^2}}, \quad (11)$$

where $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. This result is confirmed by our numerical analysis.

Fidelity of the clones can be derived by making use of unitary transformation (11) and the explicit form of $|\psi\rangle$ given by Eq. (1). After performing the calculations we derive

$$F = \frac{1 + \Lambda^2}{2} - \frac{1}{2} \sin^2 \theta (\Lambda^2 - \Lambda \sqrt{2 - 2\Lambda^2}), \quad (12)$$

where $\Lambda \equiv A/\sqrt{A^2 + 2C^2}$. For the transformation to maximize fidelity, i.e., for all $\theta \in [0, \pi]$ we impose

$$\frac{\partial F}{\partial \Lambda} = 0 \quad (13)$$

from which we get four expressions for $\Lambda(i, j=0, 1)$:

$$\Lambda_{i+2j} = (-1)^i \sqrt{\frac{1}{2} + (-1)^j \frac{\cos^2 \theta}{2\sqrt{P}}}, \quad (14)$$

where $P \equiv P(\theta) = 2 - 4 \cos^2 \theta + 3 \cos^4 \theta$ reaches the extremum values $P_{\min} = P[\arccos(\sqrt{6}/3)] = P[\pi - \arccos(\sqrt{6}/3)] = 2/3$ and $P_{\max} = P(\pi/2) = 2$. Only one of the solutions provides fidelity, which is as high as the one derived numerically. Therefore, Λ is given by

$$\Lambda \equiv \Lambda_0 \quad (15)$$

and then

$$A = \Lambda^2, \quad B = \frac{1}{2} \bar{\Lambda}^2, \quad C = \frac{1}{\sqrt{2}} \Lambda \bar{\Lambda}, \quad (16)$$

where $\bar{\Lambda} = \sqrt{1 - \Lambda^2}$. Now the unitary transformation can be written as

$$\begin{aligned} |0\rangle|0\rangle_{\text{anc}} &\rightarrow \Lambda|00\rangle|0\rangle_{\text{anc}} + \bar{\Lambda}|\psi_+\rangle|1\rangle_{\text{anc}}, \\ |1\rangle|0\rangle_{\text{anc}} &\rightarrow \Lambda|11\rangle|1\rangle_{\text{anc}} + \bar{\Lambda}|\psi_+\rangle|0\rangle_{\text{anc}} \end{aligned} \quad (17)$$

leading to the following reduced density matrices of the clones:

$$\rho_1 = \rho_2 = \begin{bmatrix} \frac{1}{2}(1 + \Lambda^2 \cos \theta) & \frac{1}{\sqrt{2}} e^{-i\phi} \Lambda \bar{\Lambda} \sin \theta \\ \frac{1}{\sqrt{2}} e^{i\phi} \Lambda \bar{\Lambda} \sin \theta & \frac{1}{2}(1 - \Lambda^2 \cos \theta) \end{bmatrix}. \quad (18)$$

Thus, fidelity of the clones, $F = F_1 = F_2$, created by this transformation can easily be calculated as

$$F = \frac{1}{2}(1 + \Lambda^2 \cos^2 \theta + \sqrt{2} \Lambda \bar{\Lambda} \sin^2 \theta). \quad (19)$$

In Fig. 2, this fidelity is depicted in comparison to fidelities for the optimal PCC and UC. As seen in Fig. 2, $F(\theta)$ has two minima $F(\arccos(\sqrt{3}/3)) = F(\pi - \arccos(\sqrt{3}/3)) = 5/6$ and a local maximum $F(\pi/2) = 1/2 + \sqrt{2}/4$. The eigenstates of Pauli σ_z matrix are cloned with the highest fidelity $F(0) = F(\pi) = 1$.

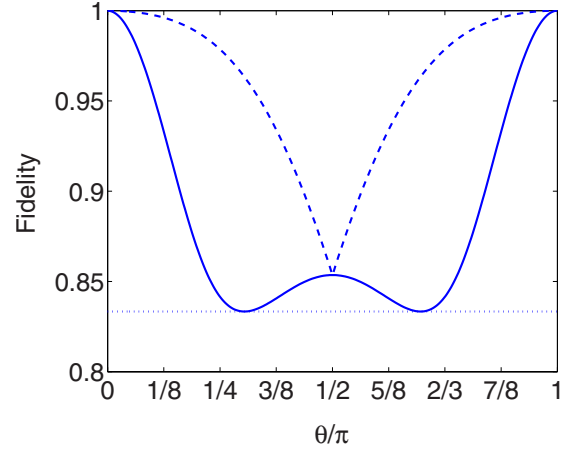


FIG. 2. (Color online) Comparison of the θ dependence of fidelities for the three optimal cloning machines: the MPCC (solid), the PCC (dashed), and the UC (dotted curve).

For better visualization of the cloned states generated by the optimal MPCC we depicted their Bloch representation in Fig. 3 in comparison to the corresponding representations for the other optimal cloning machines. Specifically, the Bloch vector $\vec{r} = (\langle \sigma_x \rangle, \langle \sigma_y \rangle, \langle \sigma_z \rangle)$ of each clone is found to be

$$\vec{r} = [\sqrt{2} \Lambda \bar{\Lambda} \cos \phi \sin \theta, \sqrt{2} \Lambda \bar{\Lambda} \sin \phi \sin \theta, \Lambda^2 \cos \theta], \quad (20)$$

which is in contrast to $\vec{r} = \frac{2}{3}[\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta]$ for the UC and to the Bloch vector

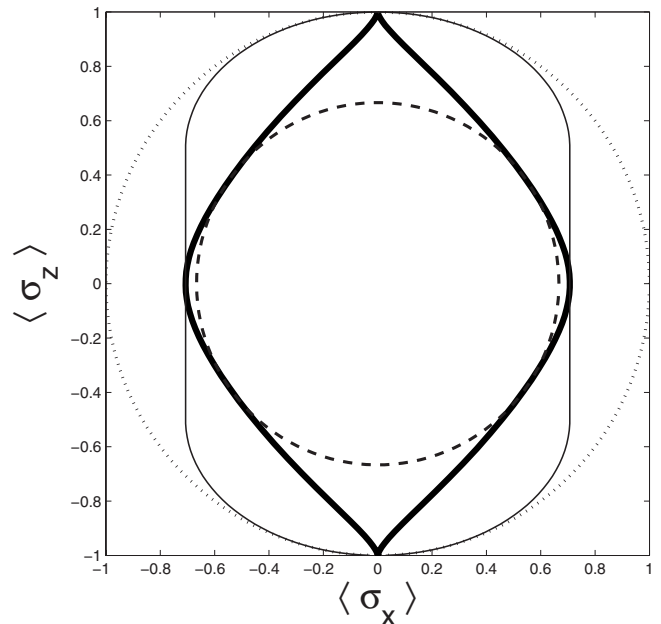


FIG. 3. Cross sections of Bloch representation for the optimal cloners: the MPCC (thick solid), the PCC (thin solid), the UC (dashed), and unphysical perfect cloner (dotted curve).

$$\vec{r} = \left[\frac{1}{\sqrt{2}} \cos \phi \sin \theta, \frac{1}{\sqrt{2}} \sin \phi \sin \theta, \frac{1}{2}(s_\theta + \cos \theta) \right] \quad (21)$$

for the optimal PCC, where $s_\theta = \text{sgn}(\pi - 2\theta)$. Actually, s_θ for $\theta = \pi/2$ can take any value in $[-1, 1]$. This is because for $\theta = \pi/2$ any linear combination of the optimal PCC transformations for south hemisphere ($|00\rangle \rightarrow |00\rangle$, $|10\rangle \rightarrow |\psi_+\rangle$) and north hemisphere ($|00\rangle \rightarrow |\psi_+\rangle$, $|10\rangle \rightarrow |11\rangle$) is also optimal [7].

IV. OPTIMALITY PROOF FOR THE MPCC

Let $\lambda = \text{Tr}_{\text{out}}(R\chi)$ be the matrix of Lagrange multipliers, then

$$\lambda = \frac{1}{4}[(1 + c_1^2)A + 2B + 2s_1^2C] \mathbb{1}_{\text{in}}. \quad (22)$$

To prove that the derived χ is the optimal TPCP map one should show that (i) $\Delta = \lambda \otimes \mathbb{1}_{\text{out}} - R$ is a positive semidefinite matrix and (ii) $\text{Tr} \lambda$ saturates the fidelity bound, i.e., $F \leq \text{Tr} \lambda$ [8]. From a technical point of view, it is more convenient to prove condition (ii) first.

Using Eqs. (12), (16), and (22) we derive

$$\begin{aligned} \text{Tr} \lambda - F &= \frac{1}{2}(1 + c_1^2)A + B + s_1^2C - \frac{1 + \Lambda^2}{2} + \frac{1}{2}s_1^2\Lambda(\Lambda - \bar{\Lambda}\sqrt{2}) \\ &= \frac{\Lambda^2}{4}[(1 + c_1)^2 + (1 - c_1)^2 - 2c_1^2 - 2] = 0. \end{aligned} \quad (23)$$

This implies that the Eq. (19) can be rewritten as

$$\lambda = \frac{F}{2} \mathbb{1}_{\text{in}}. \quad (24)$$

The eigenvalues of Δ can be written in terms of fidelity and R matrix elements as follows:

$$\begin{aligned} \delta_1 &= \frac{1}{2} \left(F - \frac{1}{2} \right), \\ \delta_2 &= \frac{1}{2} \left(F - \frac{s_1^2}{2} \right), \end{aligned}$$

$$\delta_{3,4} = \frac{1}{2} (F - R_{11} - R_{22} \pm \bar{R}), \quad (25)$$

where $\bar{R}^2 = (R_{11} - R_{22})^2 + 8R_{16}^2$. All the eigenvalues are double degenerate. Moreover, the following equation is satisfied:

$$F = R_{11} + R_{22} + \bar{R}. \quad (26)$$

Therefore, $\delta_3 = F - (3 - s_1^2)/4$ and $\delta_4 = 0$. Since $F > 3/4$ we see that $\forall_i \delta_i \geq 0$, and hence Δ is positive semidefinite. This statement completes the proof.

V. IMPLEMENTATIONS OF THE OPTIMAL MPCC

We propose two quantum circuits, shown in Figs. 4 and 5, which implement the optimal MPCC by transforming the input state $|\psi_{\text{in}}\rangle = a|000\rangle + b|100\rangle$ into

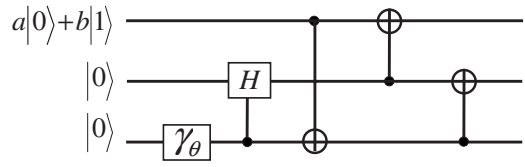


FIG. 4. A quantum circuit which implements the optimal MPCC. From left to right: rotation $R_y(\gamma)$ about y axis, controlled Hadamard gate U_{CH} , and three CNOT gates.

$$|\psi_{\text{out}}\rangle = a(\Lambda|000\rangle + \bar{\Lambda}|\psi_+\rangle|1\rangle) + b(\Lambda|111\rangle + \bar{\Lambda}|\psi_+\rangle|0\rangle). \quad (27)$$

The quantum circuit depicted in Fig. 4 performs the following transformation:

$$|\psi_{\text{out}}\rangle = U_{\text{CNOT}}^{(32)} U_{\text{CNOT}}^{(21)} U_{\text{CNOT}}^{(13)} U_{\text{CH}}^{(32)} R_y^{(3)}(\gamma_\theta) |\psi_{\text{in}}\rangle, \quad (28)$$

where the superscripts indicate qubits for which the corresponding gate is applied. The basic element of the circuit is the rotation

$$R_y(\gamma_\theta) = \begin{bmatrix} \cos(\gamma_\theta/2) & -\sin(\gamma_\theta/2) \\ \sin(\gamma_\theta/2) & \cos(\gamma_\theta/2) \end{bmatrix}, \quad (29)$$

about y axis for angle $\gamma_\theta = 2 \arccos \Lambda(\theta)$. In addition, this circuit is composed of controlled NOT (CNOT) gates, U_{CNOT} , and controlled Hadamard gate, U_{CH} , which can be decomposed as $U_{\text{CH}}^{(32)} = J^{(2)} U_{\text{CNOT}}^{(32)} J^{(2)}$, where

$$J = \frac{1}{\sqrt{4 + 2\sqrt{2}}} \begin{bmatrix} 1 & 1 + \sqrt{2} \\ 1 + \sqrt{2} & -1 \end{bmatrix}. \quad (30)$$

The basic idea behind the second quantum circuit, shown in Fig. 5, is different. The circuit acts as follows. The two CNOT gates in Fig. 5 implement the standard repetition code, which, together with the NOT gate applied to third qubit, lead to $|\psi_1\rangle = \sigma_x^{(3)} U_{\text{CNOT}}^{(13)} U_{\text{CNOT}}^{(12)} |\psi_{\text{in}}\rangle = a|001\rangle + b|110\rangle$. In our implementation, the basic gate $U(t)$ gate corresponds to the evolution operator $\exp(-iHt)$ of a system described by the interaction Hamiltonian

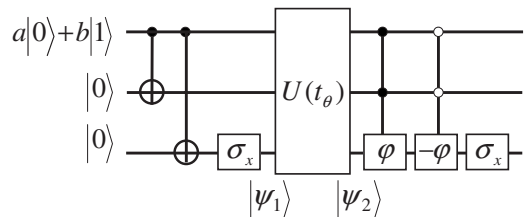


FIG. 5. Another quantum circuit implementing the optimal MPCC. The double-controlled gate with $\varphi(-\varphi)$, in the text denoted by $U_{\text{CCR}}(U_{\text{cCR}})$, corresponds to conditional rotation by a phase $\pm\varphi$, where the conditions, marked by the filled (empty) circles, are satisfied if both qubits 1 and 2 have the bits in the 1 (0) states. The first two gates are the CNOT gates, σ_x is the NOT gate, and the action of gate $U(t)$ corresponds to the evolution within time $t = t_\theta$ of the equivalent-neighbor system described by Hamiltonian (31).

$$H = \frac{\hbar\kappa}{2} \sum_{n \neq m=1}^3 \sigma_+^{(n)} \sigma_-^{(m)} + \sigma_-^{(n)} \sigma_+^{(m)}, \quad (31)$$

where κ is an effective coupling constant and $\sigma_{\pm}^{(n)} = \sigma_x^{(n)} \pm i\sigma_y^{(n)}$ are, respectively, the spin raising and lowering spin operators acting on the n th qubit. We refer to the system, described by Eq. (31), as the equivalent-neighbor model [14]. General solution of the Schrödinger equation for Hamiltonian (31) is known assuming any number of qubits and arbitrary initial conditions (see, e.g., [14]). In our special case of three qubits, one has

$$U(t)|001\rangle = C_0^{31}(t)|001\rangle + C_1^{31}(t)(|010\rangle + |100\rangle),$$

$$U(t)|110\rangle = C_0^{32}(t)|110\rangle + C_1^{32}(t)(|011\rangle + |101\rangle), \quad (32)$$

where

$$C_0^{3M}(t) = \frac{1}{3}(e^{-2i\kappa t} + 2e^{i\kappa t}),$$

$$C_1^{3M}(t) = \frac{2}{3} \sin\left(\frac{3}{2}\kappa t\right) e^{-i(i/2)(\pi + \kappa t)} \quad (33)$$

for both ‘‘excitation’’ numbers $M=1,2$. Note that $|C_0^{3M}(t)|^2 + 2|C_1^{3M}(t)|^2 = 1$. Let us choose such evolution time $t = t_{\theta}$ such that $\bar{\Lambda} = \sqrt{2}|C_2^{31}(t)|$ for a given θ . Thus, after interaction time

$$t_{\theta} = \frac{2}{3\kappa} \arcsin\left(\frac{3}{2\sqrt{2}}\bar{\Lambda}\right), \quad (34)$$

state $|\psi_1\rangle$ is transformed into

$$|\psi_2\rangle = U(t_{\theta})|\psi_1\rangle = e^{i\varphi_1}[a(\Lambda e^{i\varphi/2}|001\rangle + \bar{\Lambda}|\psi_+\rangle|0\rangle) + b(\Lambda e^{i\varphi/2}|110\rangle + \bar{\Lambda}|\psi_+\rangle|1\rangle)], \quad (35)$$

where $\varphi_k = \arg[C_k^{31}(t_{\theta})]$ ($k=0,1$) and $\varphi = 2(\varphi_0 - \varphi_1)$. The global phase factor $\exp(i\varphi_1)$ is physically irrelevant and can be dropped. While the relative phase factor $\exp(i\varphi/2)$ can be corrected by applying the double-controlled rotation gates (see Fig. 5):

$$U_{\text{CCR}}(\varphi) = I_8 + f(-\varphi)|110\rangle\langle 110| + f(\varphi)|111\rangle\langle 111|,$$

$$U_{\text{ccr}}(-\varphi) = I_8 + f(\varphi)|000\rangle\langle 000| + f(-\varphi)|001\rangle\langle 001|,$$

which generalize the standard Z-rotation gate $R(\varphi) = \text{diag}\{\exp(-i\varphi/2), \exp(i\varphi/2)\}$. In the above equations, $f(\varphi) = \exp(i\varphi/2) - 1$ and I_N is the $N \times N$ identity matrix. So finally, it is seen that $|\psi_{\text{out}}\rangle \sim \sigma_x^{(3)} U_{\text{ccr}}(-\varphi) U_{\text{CCR}}(\varphi) |\psi_2\rangle$ is the desired cloned state given by Eq. (27).

The equivalent-neighbor model of three qubits can be implemented in various ways. Here, we shortly discuss a quantum-dot model proposed by Imamoglu *et al.* [15]. The model describes interaction of N quantum-dot spins mediated by a single-mode cavity and laser fields. In our case, three semiconductor quantum dots, each with a localized single conduction-band electron, embedded inside a microdisk structure and addressed selectively by laser beams of frequency $\omega_n^{(L)}$ ($n=1,2,3$) and intensity $|E_n^{(L)}|^2$ to induce strong

coupling of the electron spins to a single cavity mode of frequency ω_{cav} . The energy spectrum of each dot is represented by three states: $|0\rangle_n$ and $|1\rangle_n$ representing the conduction-band electron spin states, while the effective state $|v\rangle_n$ describes all valence-band levels of the n th dot. It can be shown [14,15], by applying the adiabatic elimination method of the valence-band levels and the cavity field, that the system can be described by the following effective interaction Hamiltonian:

$$\hat{H}_{\text{eff}} = \frac{\hbar}{2} \sum_{n \neq m} \kappa_{nm}(t) [\hat{\sigma}_n^+ \hat{\sigma}_m^- e^{i(\Delta_n - \Delta_m)t} + \text{H.c.}], \quad (36)$$

where $\kappa_{nm}(t)$ is the effective two-dot coupling strength between the electron spins in the n th and m th dots being a function of frequencies, detunings (see Fig. 1 in [14]), intensities $|E_n^{(L)}|^2$, and the dipole coupling strengths between the levels $|0\rangle_n$ and $|1\rangle_n$, and $|v\rangle_n$. The effective Hamiltonian apparently describes direct coupling between the spins of the n th and m th dots, but it should be stressed that in the real microscopic picture the coupling between the spins is only indirect via the cavity and laser fields. To realize the equivalent-neighbor model it is enough to ensure (i) the same detunings $\Delta_n = \text{const}$ for all dots by adjusting the laser-field frequencies $\omega_n^{(L)}$ and (ii) the same effective coupling constants $\kappa_{nm}(t) = \text{const}$ by choosing the proper laser intensities, $|E_n^{(L)}|^2$ and $|E_m^{(L)}|^2$ for the adequate frequencies $\omega_n^{(L)}$ and $\omega_m^{(L)}$. Thus, Hamiltonian (36) reduces to Eq. (31). We note that other implementations (see, e.g., [16]) of the equivalent-neighbor model of interacting quantum dots can be applied here.

Imamoglu *et al.* [15] described how to perform the conditional phase flip between m th and n th dots gates, which, combined with single-qubit rotations, can be used to implement the CNOT gates. Analogously, the CCR gates can be realized in their model.

To implement of the U_{CCR} and U_{ccr} gates, we recall the well-known theorem in quantum information that any three-qubit controlled gates can be replaced by two-qubit controlled gates. Specifically, by applying lemma 6.1 of Barenco *et al.* [17], one gets $U_{\text{CCR}}(\varphi) = U_{\text{CR}}^{(13)}(\varphi/2) U_{\text{CNOT}}^{(12)} U_{\text{CR}}^{(23)}(-\varphi/2) U_{\text{CNOT}}^{(12)} U_{\text{CR}}^{(23)}(\varphi/2)$, which is given in terms of the CNOT gates and two-qubit controlled Z-rotation gates $U_{\text{CR}}(\varphi) = I_4 + f(-\varphi)|10\rangle\langle 10| + f(\varphi)|11\rangle\langle 11|$. Moreover, $U_{\text{ccr}}(-\varphi)$ can be replaced by $\sigma_x^{(1)} \sigma_x^{(2)} U_{\text{CCR}}(-\varphi) \sigma_x^{(1)} \sigma_x^{(2)}$.

VI. CONCLUSION

We proposed a $1 \rightarrow 2$ quantum cloning machine of an input qubit state assuming *a priori* information of $|\langle \sigma_z \rangle|$ (or, equivalently, $\sin \theta$). We refer to this machine as the mirror phase-covariant cloner by contrast to the standard phase-covariant cloner of an input state with *a priori* information of $\langle \sigma_z \rangle$ (or θ). We found analytical expressions describing the cloning transformation and fidelity of the clones. Applying a generalized method of Fiurášek [7], we derived and solved extremal equations for the optimal cloning transformation, which provides lower fidelity than that for the optimal phase-

covariant cloner [8]. This is because our transformation was derived assuming less knowledge about the state to be cloned. Nevertheless, our cloning machine for the whole range of θ provides fidelity higher (except $\theta = \pi/2 \pm [\pi/2 - \arccos(\sqrt{3}/3)]$) than the fidelity $F=5/6$ of the universal cloner [4,5]. The fidelity of those optimal cloning machines (see Figs. 2 and 3) can be used as thresholds for secure quantum communication and quantum teleportation [18]. Finally, we proposed quantum circuits as an implementation of

the optimal cloning transformation and suggested a physical realization in a quantum-dot model Imamoğlu *et al.* [15].

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