Computer Simulations of Entanglement Dynamics for Nonlinear System

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Abstract: In this paper we discuss the simulation method allowing modeling of quantum unitary dynamics. In particular, we discuss the model of two nonlinear oscillators (nonlinear coupler) excited by an external field as an example of systems exhibiting a possibility of entanglement generation. We show that Bell-like states are generated proving that the numerical method applied can be used as tools of quantum-mechanical simulations leading to interesting results.

Key words: computer simulations, nonlinear system, quantum-mechanical simulations, Bell-like states

I. INTRODUCTION

Quantum information theory models are one of the most interesting and widely discussed examples of those studied in quantum mechanics. Some of them are defined for quantum optical systems (various quantum optical models have been extensively discussed for instance in [1]). To describe such systems evolution very often we should apply analytical calculations requiring advanced mathematical methods that obscure the physical sense of the described phenomena. Sometimes such methods turn out to be insufficient to solve the considered problem, and for such cases we have to apply numerical calculations. The problem is to find the simplest possible method that would solve the problem without unnecessary complications. In this paper we discuss such a method that, as we shall show, can lead to interesting results despite its relative simplicity.

II. SIMULATION METHOD

This paper is devoted to the simulation of time-evolution of a quantum mechanical model describing nonlinear coupler excited by an external field. As we shall show, this model can be interesting from the point of view of the quantum information theory, since it can lead to generating maximally entangled states (Bell-like states). Since we shall concentrate on the non-damped case, we will describe the system's evolution in terms of a wave-function. Due to the discrete character of numerical calculations, it seems most natural to express this function in n-photon *Fock* basis as a vector of N complex numbers. For instance, for $N = 4$ we can write it as

$$|\psi(t)\rangle = \begin{pmatrix} C_0(t) \\ C_1(t) \\ C_2(t) \\ C_3(t) \end{pmatrix}, \tag{1}$$

where $C_i(t)$ ($i = 0, ..., N - 1$) are complex probability amplitudes corresponding to the *Fock* states $|i\rangle$ ($i = 0, ..., N - 1$), respectively. At this point one should mention the proper choice of N. Its value should guarantee that the number of the n-photon states involved in the evolution of the simulated system is considerably smaller than N.

The next step is a proper definition of the necessary quantum mechanical operators in the same basis as for the wave-function. Since the Hamiltonian of the system considered can be expressed using boson creation (annihilation) operators, we define them in the form of the following matrices (for the exemplary case $N = 4$):

$$\hat{a} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{a}^{\dagger} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}. \tag{2}$$

As we discuss the system that is not subjected to damping processes, we can describe its evolution using the unitary evolution operator $\hat{U}$. This operator is defined as

$$\hat{U} = e^{-i\hat{H}t}, \tag{3}$$

where $\hat{H}$ is the Hamiltonian governing our system (for simplicity we have expressed this definition in the units of $\hbar = 1$). Since the operator $\hat{U}$ is in the form of the matrix exponential, it could be necessary to solve the eigensystem with the Hamiltonian describing the considered model. This step can be easily done by applying standard numerical procedures [2]. Obviously, other methods of calculating matrix exponentials can be used as well. For instance, the Taylor-series expansion of the operator $\hat{U}$ can be helpful for this case. Using the evolution operator derived, we are in a position to generate the wave-function for arbitrary time t. In fact we generate the "map" of the wave-function corresponding to the moments of time separated by some period equal to t by the repeated action of the evolution operator corresponding to such time t on the initial wave-function.

For our case we assume that the system starts its evolution from the vacuum (zero-photons) state $|0\rangle$ and we act (numerically) with $\hat{U}$ on the wave-function of the system represented by the N-element vector

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4}$$

obtaining the vector representation of the desired wave-function $|\psi(\tau)\rangle$ corresponding to the state of our system for time τ:

$$|\psi(\tau)\rangle = \hat{U}^{n} |\psi(0)\rangle, \tag{5}$$

where $\tau = nt$ (n = 1, 2, ...). Obviously, this method does not limit our considerations to the *vacuum* state only. For instance, if we consider an n-photon state as the initial one, we put 1 on the (n + 1)th position in the vector (4). For other quantum states we can use their expansion in the *Fock* basis, for example for the Glauber coherent state $|\alpha\rangle$ [3] we have

$$|\psi\rangle = |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{6}$$

where α is equal to the mean number of photons $\langle\hat{n}\rangle = \langle\hat{a}^{\dagger}\hat{a}\rangle$ for this state.

This method has already been applied successfully, for instance for computer simulations of the evolution of quantum nonlinear oscillators generating n-photon *Fock* states [4] or *finite-dimensional coherent states* [5]. However, one should keep in mind that this paper deals with entanglement dynamics. Therefore, we should extend our considerations to the case of a two-mode system (the entanglement for the exemplary model discussed here will be built between the states corresponding to the modes of two interacting nonlinear oscillators). In consequence, we should define the additional operators $\hat{b}(\hat{b}^{\dagger})$ analogously to those already defined – $\hat{a}(\hat{a}^{\dagger})$, and next extend our considerations from the Hilbert space $\mathcal{H}$ defined for one oscillator to the product of such spaces corresponding to the two oscillators $\mathcal{H}_a \otimes \mathcal{H}_b$. Obviously, for such a case the wave-function describing the system's evolution will be represented by the product of the vectors $|\psi(t)\rangle_a \otimes |\psi(t)\rangle_b$, where the indexes {$a$, b} correspond to the modes of oscillators a and b, respectively.

III. THE MODEL AND ITS EVOLUTION

An exemplary model for which the entanglement is generated is that of a nonlinear coupler comprising two interacting nonlinear oscillators. In addition, we assume that the coupler is externally driven to be a coherent field in one of the oscillator modes (in this case it is the mode a). The Hamiltonian describing such a system can be expressed as:

$$\hat{H} = \frac{\chi_a}{2}\left(\hat{a}^{\dagger}\right)^2 \hat{a}^2 + \frac{\chi_b}{2}\left(\hat{b}^{\dagger}\right)^2 \hat{b}^2 + \epsilon\left(\hat{a}^{\dagger}\right)^2 \hat{b}^2 + \\ + \epsilon^{*}\left(\hat{b}^{\dagger}\right)^2 \hat{a}^2 + \alpha\hat{a}^{\dagger} + \alpha^{*}\hat{a}, \tag{7}$$

where $\hat{a}(\hat{a}^{\dagger})$ and $\hat{b}(\hat{b}^{\dagger})$ are the usual photon annihilation (creation) operators in the oscillators modes a and b, respectively, $\chi_a(\chi_b)$ describe the nonlinearities of the oscillators, α is the coupling constant between the external coherent field and the mode *a*, and finally ϵ is the coupling constant describing the interaction between the two oscillators.

In this paper we assume that the system was initially in the state $|2\rangle_a|0\rangle_b$, i.e. we have two photons in mode *a* and no photons in *b*. Applying the method described above we get the probability amplitudes $c_{i,j}$ for the states $|i\rangle|j\rangle$ (i, j = 0, 1, ..., *k*). However, since for the discussed model we can observe some resonances between the levels spacing for an unperturbed system and the external excitations, we can practically neglect the influence of the states corresponding to $k = 4$ and higher. In fact, when the external excitation is assumed to be sufficiently weak, *i.e.* $\alpha \ll \chi_i$ ($i = \{1, 2\}$) these states remain practically unpopulated – for the values of the parameters chosen here the probabilities corresponding to $k \geq 4$ are of the order of $\sim 10^{-4}$ and smaller. Consequently, one can say that the system's evolution is closed within the set of three states and hence, the wave-function can be expressed as

$$|\psi(t)\rangle_{cut} = c_{2,0}(t)|2\rangle_a|0\rangle_b + \\ + c_{1,2}(t)|1\rangle_a|2\rangle_b + c_{0,2}(t)|0\rangle_a|2\rangle_b . \tag{8}$$

Due to this fact the system discussed can be referred as to as *quantum nonlinear scissors* [6].

The results of our numerical calculations are depicted in Fig. 1 showing the probabilities $|c_{i,j}|^2$ ($i = \{0, 1, 2\}$ and $j = \{0, 2\}$) corresponding to the three states mentioned. We have assumed that external coupling is weak $-\epsilon \ll \chi_{a,b}$ ($\epsilon = \pi/25$, $\chi_{a,b} = 25$, and additionally $\epsilon = 2\alpha/3$). We see that these probabilities change their values periodically from zero to unity (for $|2\rangle_a|0\rangle_b$), to ~0.9 (for $|0\rangle_a|2\rangle_b$) and finally, to ~0.65 for the state $|1\rangle_a|2\rangle_b$. What is most interesting, the lines corresponding to the probabilities for the states $|2\rangle_a|0\rangle_b$ and $|0\rangle_a|2\rangle_b$ intersects at the points where they are equal to 0.5. In fact, for such moments of time, Bell-like states are generated, particularly there are the states:

$$|B_1\rangle = \frac{1}{\sqrt{2}}(|2\rangle_a|0\rangle_b + i|0\rangle_a|2\rangle_b), \tag{9a}$$

$$|B_2\rangle = \frac{1}{\sqrt{2}}(|2\rangle_a|0\rangle_b - i|0\rangle_a|2\rangle_b). \tag{9b}$$

These states are the maximally entangled states defined for the *qubit-qubit* subsystem. However, one should keep in mind that our system behaves as a *qubit-qutrit* one, because of the fact that for mode *a* three states are involved. Therefore, we could expect that other Bell-like states can be generated as well. In fact, we can observe such a situation when the probability corresponding to the state $|1\rangle_a|2\rangle_b$ reaches its maximal value. For this case another Bell-like state is produced, *i.e.* our system is in the state

$$|B_3\rangle = \frac{1}{\sqrt{2}}(|2\rangle_a|0\rangle_b + |1\rangle_a|2\rangle_b). \tag{10}$$

What should be noted is that for this case we get almost a maximally entangled state, too.

To check whether (and how exactly) we generate Bell-like states, we can calculate the fidelities between the state of the wave-function describing the system's evolution and each of the above states – the case of $|B_3\rangle$ generation is not so obvious. This fidelity we define as

$$\mathcal{F}(t) = |\langle\psi(t)|B_i\rangle|^2 , \qquad i = \{1, 2, 3\} \tag{11}$$

and from Fig. 2 we can easily see that indeed, all three mentioned here Bel-llike states are generated almost perfectly.

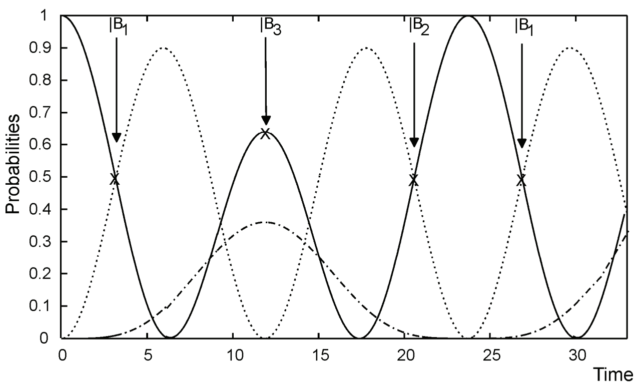

Fig. 1. The probabilities corresponding to the states $|2\rangle_a|0\rangle_b$ (solid line) $|0\rangle_a|2\rangle_b$) (dashed line), and $|1\rangle_a|2\rangle_b$ (dashed-dotted line). The cross-marks correspond to Bell states generation. The coupling strengths $\epsilon = \pi\alpha/25 = 2\epsilon/3$, and the nonlinearities $\chi_a = \chi_b = 25$. Time is scaled in the units of $1/\chi$

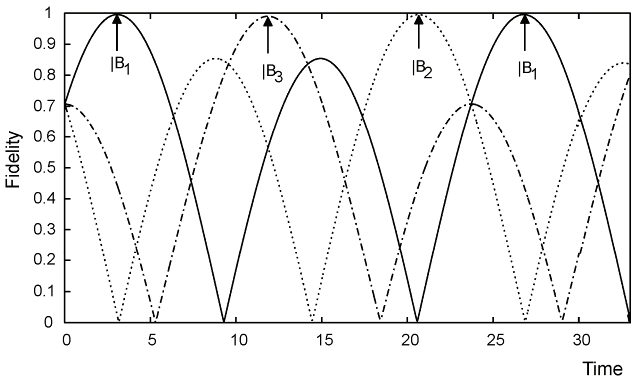

Fig. 2. The fidelities corresponding to the Bell states: $|B_1\rangle$ – solid line, $|B_2\rangle$ – dashed line and $|B_3\rangle$ – dashed-dotted line. Arrows point to the moments of time when the Bell states are generated. All parameters are the same as those for Fig. 1.

In this paper we have concentrated on the non-damping case. If we want to include the damping phenomena, we can for instance apply the master equation approach, quantum trajectories or quantum-state diffusion methods (description and comparison of these three methods can be found in [7]). However, one should keep in mind that for such cases we need much more computing power than for the case discussed here. For the computer simulations of the damped systems the requirements concerning both memory and processor use grow enormously with increasing dimension of the problem.

IV. SUMMARY

In this paper we have concentrated on the computer simulation of the time-evolution of the quantum nonlinear coupler excited by an external coherent field. Since we dealt with the non-damping case this evolution was modeled by application of the unitary evolution operator on the wave-function describing our system. We have shown that the evolution can be modeled effectively and it exhibits interesting phenomena. In particular, the maximally entangled states are generated and we have shown that these states are Bell-like ones.

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