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Paradoxes of measures of quantum entanglement and Bell's inequality violation in two-qubit systems

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Abstract. We review some counterintuitive properties of standard measures describing quantum entanglement and violation of Bell's inequality (often referred to as "nonlocality") in two-qubit systems. By comparing the nonlocality, negativity, concurrence, and relative entropy of entanglement, we show: (i) ambiguity in ordering states with the entanglement measures, (ii) ambiguity of robustness of entanglement in lossy systems and (iii) existence of two-qubit mixed states more entangled than pure states having the same negativity or nonlocality. To support our conclusions, we performed a Monte Carlo simulation of 10^6 two-qubit states and calculated all the entanglement measures for them. Our demonstration of the relativity of entanglement measures implies also how desirable is to properly use an operationally-defined entanglement measure rather than to apply formally-defined standard measures. In fact, the problem of estimating the degree of entanglement of a bipartite system cannot be analyzed separately from the measurement process that changes the system and from the intended application of the generated entanglement.

1. Introduction

Quantum entanglement [1,2], being at heart of Bell's theorem [3], is considered to be an essential resource for quantum engineering, quantum communication, quantum computation, and quantum information [4]. There were proposed various entanglement measures and criteria to detect entanglement. Nevertheless, despite the impressive progress in understanding this phenomenon (see a recent comprehensive review by Horodecki et al. [5] and references therein), a complete theory of quantum entanglement has not been developed yet.

It is a commonly accepted fact that the entropy of entanglement of two systems, which is defined to be the von Neumann entropy of one of the systems, is the unique entanglement measure for bipartite systems in a pure state [6]. However, in the case of two systems in a mixed state, there is no unique entanglement measure. In order to describe properties of quantum entanglement of bipartite systems various measures have been proposed. Examples include [5]: entanglement of formation, distillable entanglement, entanglement cost, PPT entanglement cost, relative entropy of entanglement, or geometrical measures of entanglement.

It should be stressed that classification of entanglement measures of mixed states and effective methods of calculation of such measures are among the most important but still underdeveloped (with a few exceptions) problems of quantum information [7].

Here, we shortly review counterintuitive properties of some entanglement measures in the simplest non-trivial case of entanglement of two qubits.

2. Measures of quantum entanglement

We will study quantum entanglement and closely related violation of Bell's inequality for two qubits in mixed states according to some standard measures:

(i) To describe the entanglement of formation [8] of a given two-qubit state $\hat{\rho}$, we apply the Wootters concurrence [9] defined as

$$C(\hat{\rho}) = \max\left(0, 2\max_i \lambda_i - \sum_i \lambda_i\right) \tag{1}$$

in terms of λ_i's, which are the square roots of the eigenvalues of $\hat{\rho}(\hat{\sigma}_2 \otimes \hat{\sigma}_2)\hat{\rho}^*(\hat{\sigma}_2 \otimes \hat{\sigma}_2)$, where σ_2 is the Pauli spin matrix and asterisk stands for complex conjugation. The concurrence $C(\hat{\rho})$ is related to the entanglement of formation, $E_F(\hat{\rho})$, as follows [9]:

$$E_F(\hat{\rho}) = \mathcal{W}[C(\hat{\rho})], \quad \text{where} \quad \mathcal{W}(x) \equiv h\left(\frac{1}{2}[1+\sqrt{1-x^2}]\right), \tag{2}$$

and $h(y) = -y\lg y - (1-y)\lg(1-y)$ is binary entropy.

(ii) The PPT entanglement cost, which is the entanglement cost [5] under operations preserving positivity of the partial transposition (PPT), can be given as [10,11]:

$$E_{\mathrm{PPT}}(\hat{\rho}) = \lg[N(\hat{\rho}) + 1] \tag{3}$$

in terms of the negativity:

$$N(\hat{\rho}) = 2\sum_j \max(0, -\mu_j). \tag{4}$$

These measures are related to the Peres-Horodecki criterion [12,13]. In Eq. (4), μ_j are the eigenvalues of the partial transpose $\hat{\rho}^{\Gamma}$.

(iii) The relative entropy of entanglement (REE) [14,15] is a measure of entanglement corresponding to a "distance" of an entangled state from separable states. Precisely, the REE can be defined as the minimum of the relative quantum entropy

$$S(\hat{\rho}||\hat{\rho}_{\mathrm{sep}}) = \mathrm{Tr}\,(\hat{\rho}\lg\hat{\rho} - \hat{\rho}\lg\hat{\rho}_{\mathrm{sep}}) \tag{5}$$

in the set $\mathcal{D}$ of all separable states $\hat{\rho}_{\mathrm{sep}}$, i.e.,

$$E_R(\hat{\rho}) = \min_{\hat{\rho}_{\mathrm{sep}}\in\mathcal{D}} S(\hat{\rho}||\hat{\rho}_{\mathrm{sep}}) \equiv S(\hat{\rho}||\hat{\rho}_{\mathrm{css}}), \tag{6}$$

where $\hat{\rho}_{\mathrm{css}}$ denotes the closest separable state (CSS) to $\hat{\rho}$. Numerical problems to calculate the REE are shortly discussed in Appendix A.

(iv) To describe a degree of violation of Bell's inequality [3] due to Clauser, Horne, Shimony and Holt (CHSH) [16], we use the modified Horodecki measure [17,19]:

$$B(\hat{\rho}) \equiv \sqrt{\max\left[0, \max_{j<k}(u_j + u_k) - 1\right]}, \tag{7}$$

which is given in terms of the eigenvalues u_j $(j = 1,2,3)$ of $U_{\hat{\rho}} = T_{\hat{\rho}}^T T_{\hat{\rho}}$, where $T_{\hat{\rho}}$ is a real matrix with elements $t_{nm} = \mathrm{Tr}\,[\hat{\rho}\,(\hat{\sigma}_n \otimes \hat{\sigma}_m)]$, $T_{\hat{\rho}}^T$ is the transposition of $T_{\hat{\rho}}$ and $\hat{\sigma}_n$ $(n = 1,2,3)$ are Pauli's spin matrices. For short, we refer to B as "nonlocality" (measure).

For any two-qubit pure state $|\psi\rangle$, the nonlocality B is equal to the entanglement measures C and N:

$$B(|\psi\rangle) = C(|\psi\rangle) = N(|\psi\rangle). \tag{8}$$

It is seen that for this case the measures B, C and N correspond to the relative entropy of entanglement E_R and von Neumann's entropy:

$$\mathcal{W}[B(|\psi\rangle)] = \mathcal{W}[C(|\psi\rangle)] = \mathcal{W}[N(|\psi\rangle)] = E_R(|\psi\rangle) = E_{\mathrm{Neumann}}(|\psi\rangle), \tag{9}$$

where $\mathcal{W}$ is given in Eq. (2).

In the following we describe somewhat surprising properties of the entanglement measures for two-qubits in *mixed* states. For brevity, by referring to the entanglement measures, we also mean the nonlocality B.

3. Ambiguity in ordering states with entanglement measures

Our problem can be posed as follows:

Problem 1. *Two measures of entanglement, say $\mathcal{E}'$ and $\mathcal{E}''$, imply the same ordering of states if the condition [18]*

$$\mathcal{E}'(\hat{\rho}_1) < \mathcal{E}'(\hat{\rho}_2) \Leftrightarrow \mathcal{E}''(\hat{\rho}_1) < \mathcal{E}''(\hat{\rho}_2) \tag{10}$$

is satisfied for arbitrary states $\hat{\rho}_1$ and $\hat{\rho}_2$. The question is whether this condition is fulfilled for all "good" entanglement measures.

In early fundamental works on quantum information, it is often claimed that good entanglement measures should fulfill this condition. For example, in Ref. [14] it was stated that: "For consistency, it is only important that if $\hat{\rho}_1$ is more entangled then $\hat{\rho}_2$ for one measure than it also must be for all other measures."

For qubits in pure states, condition (10) is always fulfilled, since all good measures are equivalent. However, standard measures can imply different ordering of mixed states even for only two qubits. This was first shown numerically by Eisert and Plenio [18] by analyzing their results of Monte Carlo simulations of two-qubit states. The problem was then analyzed by others [19–28].

To our knowledge, the first analytical examples of two-qubit states violating condition (10) were given in Refs. [19,23]. In Ref. [24], to find analytical examples of extreme violation of Eq. (10), we applied the results of Verstraete et al. [29] concerning allowed values of the negativity N for a given value of the concurrence C.

Note that the violation of condition (10) cannot be observed for pure states of two-qubit systems. By contrast, for three-level systems (the so-called qutrits), analytical examples of violation of the condition are known even for pure states [20–22].

The property that ordering of states depends on the applied entanglement measure sounds counter-intuitive. Nevertheless, it is physically sound since states, which are differently ordered according to two measures, cannot be transformed into each other with 100% efficiency by applying local quantum operations and classical communication (LOCC) only. Virmani and Plenio [21] proved in general terms that all good asymptotic entanglement measures are either identical or have to imply different ordering on some quantum states.

In Ref. [25], the three measures (the negativity, concurrence, and REE) were compared and found analytical examples of states (say $\hat{\rho}'$ and $\hat{\rho}''$) for which one measure implies state ordering opposite to that implied by the other two measures:

$$\begin{aligned} &C(\hat{\rho}') < C(\hat{\rho}''), \quad N(\hat{\rho}') < N(\hat{\rho}''), \quad E_R(\hat{\rho}') > E_R(\hat{\rho}''); \\ &C(\hat{\rho}') < C(\hat{\rho}''), \quad N(\hat{\rho}') > N(\hat{\rho}''), \quad E_R(\hat{\rho}') < E_R(\hat{\rho}''); \\ &C(\hat{\rho}') > C(\hat{\rho}''), \quad N(\hat{\rho}') < N(\hat{\rho}''), \quad E_R(\hat{\rho}') < E_R(\hat{\rho}''). \end{aligned} \tag{11}$$

There can be found other analytical examples of states exhibiting even more peculiar ordering of states according to these three measures. Examples include pairs of states for which a degree of entanglement is preserved according to one or two measures but it is different according to the other measures, e.g.:

$$\begin{aligned} &C(\hat{\rho}') = C(\hat{\rho}''), \quad N(\hat{\rho}') < N(\hat{\rho}''), \quad E_R(\hat{\rho}') > E_R(\hat{\rho}''); \\ &C(\hat{\rho}') < C(\hat{\rho}''), \quad N(\hat{\rho}') = N(\hat{\rho}''), \quad E_R(\hat{\rho}') > E_R(\hat{\rho}''); \\ &C(\hat{\rho}') < C(\hat{\rho}''), \quad N(\hat{\rho}') > N(\hat{\rho}''), \quad E_R(\hat{\rho}') = E_R(\hat{\rho}''). \end{aligned} \tag{12}$$

and

$$\begin{aligned} &C(\hat{\rho}') = C(\hat{\rho}''), \quad N(\hat{\rho}') = N(\hat{\rho}''), \quad E_R(\hat{\rho}') < E_R(\hat{\rho}''); \\ &C(\hat{\rho}') = C(\hat{\rho}''), \quad N(\hat{\rho}') < N(\hat{\rho}''), \quad E_R(\hat{\rho}') = E_R(\hat{\rho}''); \\ &C(\hat{\rho}') < C(\hat{\rho}''), \quad N(\hat{\rho}') = N(\hat{\rho}''), \quad E_R(\hat{\rho}') = E_R(\hat{\rho}''). \end{aligned} \tag{13}$$

The comparative analyses presented in Refs. [19,23–25] are not only related to a mathematical problem of classification of states according to various entanglement measures. They could also enable a deeper understanding of some physical aspects of entanglement.

3.1. Nonequivalent states with the same entanglement according to E_R, C and N

Problem 2. *Find analytical examples of *nonequivalent* two-qubit states $\hat{\rho}'$ and $\hat{\rho}''$ exhibiting the same entanglement of formation [$C(\hat{\rho}') = C(\hat{\rho}'')$], the same PPT entanglement cost [$N(\hat{\rho}') = N(\hat{\rho}'')$], and the same relative entropy of entanglement [$E_R(\hat{\rho}') = E_R(\hat{\rho}'')$]?*

As a first attempt to find such an example, let us compare two different pure states:

$$\begin{aligned} |\psi'\rangle &= c'_{00}|00\rangle + c'_{01}|01\rangle + c'_{10}|10\rangle + c'_{11}|11\rangle, \\ |\psi''\rangle &= c''_{00}|00\rangle + c''_{01}|01\rangle + c''_{10}|10\rangle + c''_{11}|11\rangle, \end{aligned} \tag{14}$$

fulfilling the condition

$$|c'_{00}c'_{11} - c'_{01}c'_{10}| = |c''_{00}c''_{11} - c''_{01}c''_{10}|, \tag{15}$$

which guarantees the same degree of entanglement according to the measures C, N and E_R. However, states $|\psi'\rangle$ and $|\psi''\rangle$ can be transformed into each other by local operations. Namely, by applying local rotations, $|\psi\rangle$ can be converted into ($p = p', p''$)

$$|\tilde{\psi}(p)\rangle = \sqrt{p}|01\rangle + \sqrt{1-p}|10\rangle \tag{16}$$

for which the negativity and concurrence are equal to $2\sqrt{p(1-p)}$. The same value is obtained also for $|\tilde{\psi}(1-p)\rangle$, but this state can be transformed into $|\tilde{\psi}(p)\rangle$ by applying the NOT gate to each of the qubits. This shows that pure states are *not* a good example of states satisfying the conditions specified in Problem 2.

As a second attempt, let us compare two Bell diagonal states described by $\hat{\rho}'_B$ and $\hat{\rho}''_B$ with the same maximum eigenvalue $\max_i \lambda_i > 1/2$. These states have the same entanglement according to the measures C, N and E_R. However, as shown in Ref. [25], states $\hat{\rho}'_B$ and $\hat{\rho}''_B$ exhibit different nonlocality, i.e., violate Bell's inequality to different degree. Specifically, the nonlocality B for a Bell diagonal state is given by [25]:

$$B(\hat{\rho}_B) = \sqrt{\max\{0, 2 \max_{(i,j,k)}[(\lambda_i - \lambda_j)^2 + (\lambda_k - \lambda_4)^2] - 1\}}, \tag{17}$$

where subscripts (i,j,k) correspond to cyclic permutations of $(1,2,3)$. It is seen that the violation of Bell's inequality depends on all values of λ_i, while the entanglement measures E_R, C, and N depend only on the largest value $\max_i \lambda_i > 1/2$. Thus, states $\hat{\rho}'_B$ and $\hat{\rho}''_B$, fulfilling the conditions $\mathrm{eig}(\hat{\rho}'_B) \neq \mathrm{eig}(\hat{\rho}''_B)$ and $\max\{\mathrm{eig}(\hat{\rho}'_B)\} = \max\{\mathrm{eig}(\hat{\rho}''_B)\} > 1/2$, have the same entanglement measures: $E_R(\hat{\rho}'_B) = E_R(\hat{\rho}''_B)$, $C(\hat{\rho}'_B) = C(\hat{\rho}''_B)$ and $N(\hat{\rho}'_B) = N(\hat{\rho}''_B)$, but the states are not equivalent as they exhibit different nonlocality, $B(\hat{\rho}'_B) \neq B(\hat{\rho}''_B)$.

4. Ambiguity of robustness of entanglement

4.1. Maximally entangled pure states in lossy cavities

Let us analyze the following problem:

Problem 3. *Which maximally entangled pure states are the most fragile or robust to decoherence of two qubits in lossy cavities?*

This problem was addressed in Refs. [19,23] by analyzing decoherence of optical photon-number qubits stored initially in the following three maximally entangled (pure) states (MES):

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \quad |\Psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{18}$$

$$|\Psi_3\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) \equiv \frac{1}{\sqrt{2}}(|0,+\rangle + |1,-\rangle), \tag{19}$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. State $|\Psi_3\rangle$ can be obtained from $|\Psi_2\rangle$ by applying Hadamard's gate to the second qubit.

To address Problem 3, let us analyze two entangled qubits in a superposition of vacuum and single-photon states (so-called photon-number qubits) in a lossy cavity (or, equivalently, in two cavities). Then, one can apply the standard master-equation approach to describe the effect of radiative decay of cavities (i.e., zero-temperature reservoirs) on entanglement of two qubits according to the concurrence $C_k(t)$, negativity $N_k(t)$, and nonlocality $B_k(t)$ [19]. In Fig. 1, it is assumed that the qubits are initially in the

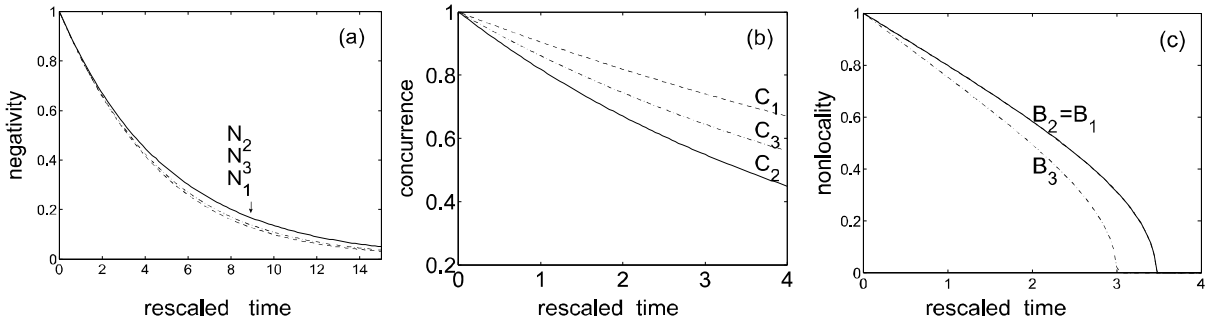

Fig. 1. Decay of entanglement between two qubits initially in the maximally entangled states $|\Psi_k\rangle$ (for k = 1,2,3) in lossy cavities with damping rates $\gamma = 0.1$ described by: (a) the negativity N, (b) the concurrence C, and (c) the nonlocality B. It is seen that there is no simple answer to the question which of the initial states $|\Psi_k\rangle$ is the most fragile (or robust) to decoherence. In the discussed model of dissipation, the fastest decoherence exhibits: $|\Psi_1\rangle$ according to N, $|\Psi_2\rangle$ according to C, and $|\Psi_3\rangle$ according to B.

MES $|\Psi_k\rangle$ for k = 1,2,3 and the cavity damping rate is $\gamma = 0.1$. By analyzing Fig. 1, one can conclude that entanglement decays in this model fulfill the inequalities:

$$\begin{aligned} N_2(t) &\geqslant N_3(t) \;\geqslant\; N_1(t), \\ B_1(t) = B_2(t) &\;\geqslant\; B_3(t), \\ C_1(t) &\geqslant C_3(t) \;\geqslant\; C_2(t). \end{aligned} \tag{20}$$

It is worth noting that due to the Markov approximation assumed in the derivation of the master equation, our conclusions are valid for evolution times t short in comparison to reservoir decay time γ^{-1}, and much longer than correlation time τ_c of reservoir(s), i.e., $\tau_c \ll t - t_0 \ll \gamma_j^{-1}$, where t_0 is the initial evolution time. Thus, in this specific dissipation model, the most fragile to dissipation is $|\Psi_1\rangle$ according to the negativity N, $|\Psi_2\rangle$ according to the concurrence C, and $|\Psi_3\rangle$ according to the nonlocality B. The results seem to be contradicting, but it should be remembered that measures C, N and B describe different aspects of mixed states even if for pure states they coincide $C = N = B$. Results of Refs. [19,23] clearly confirm the relativity of state ordering by C, N and B. This example of Ref. [19] was probably the first demonstration of this property in a real physical process.

4.2. Maximally entangled mixed states in lossy cavities

Here, we analyze decay of Werner's state, which can be defined for $p \in \langle 0, 1\rangle$ as [31]:

$$\hat{\rho}_1^{(p)}(0) = p|\Psi_1\rangle\langle\Psi_1| + \tfrac{1-p}{4}\,\hat{I}\otimes\hat{I}, \tag{21}$$

which is a mixture of the singlet state, $|\Psi_1\rangle$, and maximally mixed state, given by $\hat{I}\otimes\hat{I}$, where $\hat{I}$ is identity operator. Original Werner's state can be generalized for mixtures of other Bell states with $\hat{I}\otimes\hat{I}$. Thus, one can define Werner-type state as follows ($k = 2, 3$):

$$\hat{\rho}_k^{(p)}(0) = p|\Psi_k\rangle\langle\Psi_k| + \tfrac{1-p}{4}\,\hat{I}\otimes\hat{I}, \tag{22}$$

where $|\Psi_2\rangle$ and $|\Psi_3\rangle$ are given by Eqs (18) and (19), respectively.

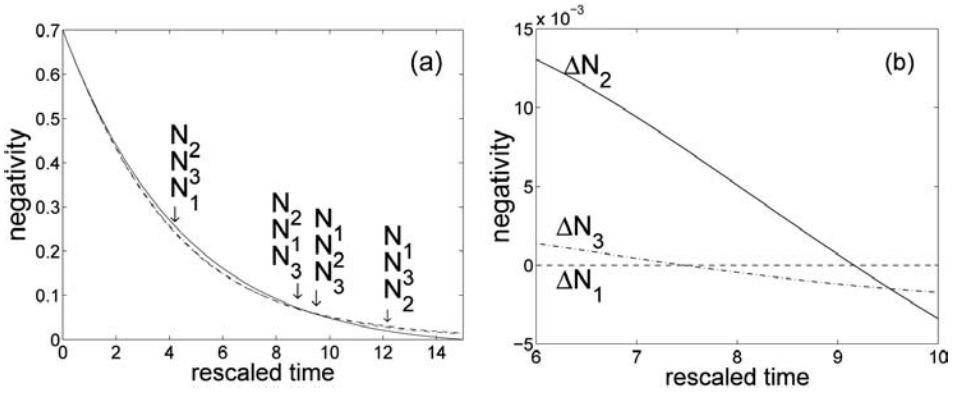

Fig. 2. Decay of entanglement of two photon-number qubits in a lossy cavity. Entanglement is measured by the negativity (a) N_k and (b) $\Delta N_k = N_k - N_1$ for qubits initially in Werner's states $\hat{\rho}_k^{(p)}(0)$ for $k = 1,2,3$ and $p = 0.8$. The cavities damping rate is $\gamma = 0.1$. For clarity, the scale of figure (b) is enlarged in comparison to figure (a).

Werner's states can be considered as *maximally entangled mixed states* (MEMS) of two qubits since the amount of entanglement of these states cannot be increased by any unitary transformation [32] and they are maximally entangled (according to the concurrence) for a given value of linear entropy [33].

Let us ask more specific question related to Problem 3:

Problem 4. *Which MEMS are the most robust to dissipation in the discussed model of lossy cavities?*

Even for such formulated question there is no simple answer. To show this we analyze the same model of decaying photon-number qubits in a lossy cavity (or cavities) as studied in Sect. 4.1, but for qubits initially in Werner's states $\hat{\rho}_k^{(p)}(0)$ for $k = 1,2,3$ and $p = 0.8$. Let us compare the decays of the negativity as shown in Fig. 2 and also described in detail in Table I in Ref. [19]. It is seen that a given Werner state can be more robust to decay than another Werner's state at short evolution times but, in turn, less robust at longer times. The differences between the negativity values for various states shown in Fig. 2 are not very large but still distinct.

5. Mixed states more entangled than pure states

Problem 5. *Can two-qubit *mixed* states be more entangled than *pure* states according to some entanglement measure $\mathcal{E}'$ at a fixed value of another entanglement measure $\mathcal{E}''$ assuming $\mathcal{E}'(\hat{\rho}) \leqslant \mathcal{E}''(\hat{\rho})$ for any state $\hat{\rho}$?*

It can be shown analytically that pure states are the upper bound for the negativity for a given value of the concurrence [29], as shown in Fig. 3(a), and the upper bound for the REE as a function of the concurrence [15], as presented in Fig. 3(b). Similar conclusions can be drawn for, e.g., the nonlocality for a given value of the concurrence [see Fig. 3(c)], and the nonlocality as a function of the negativity.

Thus, it is reasonable to conjecture that pure states are the upper bound also for the REE, e.g., for a given value of the negativity. But it was shown in Refs. [25,28] that this conjecture is wrong [see Fig. 3(e)]. This property can be demonstrated analytically on the example of, e.g., the Horodecki state [5] defined as a mixture of the maximally entangled state [e.g., the singlet state $|\Psi_1\rangle$] and a separable state

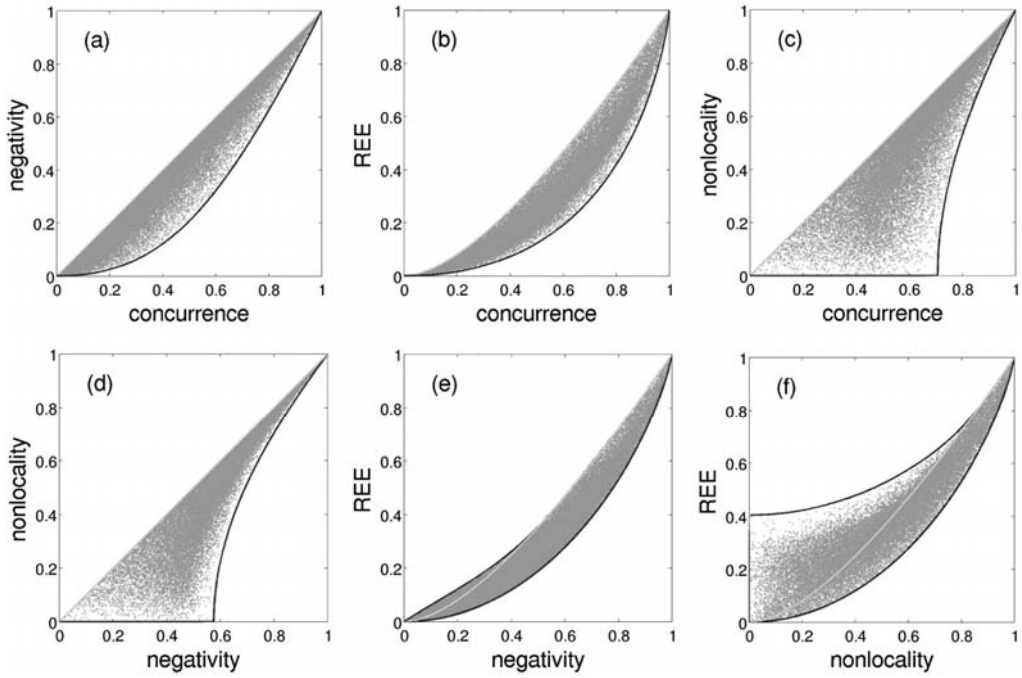

Fig. 3. (Color online) Entanglement and nonlocality measures for 10^6 two-qubit states $\hat{\rho}$ generated by a Monte Carlo simulation. Green curves correspond to pure states, and blue curves show the upper and lower bounds of a one measure $\mathcal{E}'(\hat{\rho})$ as a function of another $\mathcal{E}''(\hat{\rho})$. It is seen that pure states $|\psi\rangle$ lie for the whole range of abscissa at the upper bound of: (a) the negativity $N(\hat{\rho})$ for a given value of the concurrence $C(\hat{\rho})$ (b) the REE $E_R(\hat{\rho})$ vs $C(\hat{\rho})$, (c) the nonlocality $B(\hat{\rho})$ vs $C(\hat{\rho})$, and (d) $B(\hat{\rho})$ vs $N(\hat{\rho})$. However, for (e) $E_R(\hat{\rho})$ vs $N(\hat{\rho})$ and (e) $E_R(\hat{\rho})$ vs $B(\hat{\rho})$ pure states are at the upper bound for abscissa values close to one only. Thus, in the cases (e) and (f), the entanglement of mixed states can exceed that of pure states for abscissa values close to zero.

orthogonal to it (e.g., $|00\rangle$):

$$\hat{\rho}^{(H)} = p|\Psi_1\rangle\langle\Psi_1| + (1-p)|00\rangle\langle 00|, \tag{23}$$

where $p \in \langle 0, 1\rangle$. The negativity and REE for the Horodecki state are equal to

$$N(\hat{\rho}^{(H)}) = \sqrt{(1-p)^2 + p^2} - (1-p), \tag{24}$$

$$E_R(\hat{\rho}^{(H)}) \equiv E_R^{(H)}(N) = 2h(1-p/2) - h(p) - p, \tag{25}$$

respectively, where $p = \sqrt{2N(1+N)} - N$ and $h(x)$ is binary entropy. By comparing the REEs for Horodecki's state and for pure states, it can be shown that [25,28]:

$$E_R^{(H)}(N) > E_R^{(P)}(N) \quad \text{for } 0 < N < N_Y, \tag{26}$$

$$E_R^{(H)}(N) < E_R^{(P)}(N) \quad \text{for } N_Y < N < 1, \tag{27}$$

where $N_Y = 0.3770\cdots$ and $E_R^{(H)}(N_Y) = E_R^{(P)}(N_Y) = 0.2279\cdots$, which corresponds to point Y in Fig. 4. These inequalities were shown analytically by expending $E_R^{(H)}(N)$ and $E_R^{(P)}(N)$ in power series of $N = \epsilon$ ($N = 1 - \epsilon$) for values close to 0 (1). Moreover, mixed states corresponding to blue region

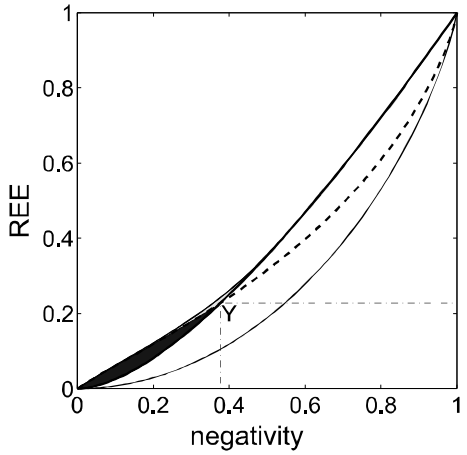

Fig. 4. (Color online) The relative entropy of entanglement (REE), $E_R(\hat{\rho})$, as a function of the negativity, $N(\hat{\rho})$, for pure states (thick solid curve), Horodecki states (dashed curve) and Bell diagonal states (thin solid curve). Blue and yellow regions correspond to mixed states with the REE higher than that for pure states for a given value the negativity. States in blue region are described in the text.

in Fig. 4, for which the inequality in Eq. (26) holds, can be obtained by mixing the Horodecki state $\hat{\rho}_H$ with a separable state $\hat{\rho}_{\rm css}^{(H)}$ closest to $\hat{\rho}_H$ [25]:

$$\hat{\rho}^{(H')}(p,N) = (1-x)\hat{\rho}^{(H)} + x\hat{\rho}_{\rm css}^{(H)}, \tag{28}$$

where $N \in \langle 0,1\rangle$, $p \in \langle \sqrt{2N(1+N)} - N, 1\rangle$ and $x = [(N+p)^2 - 2N(1+N)]/[p^2(1+N)]$. The closest separable state $\hat{\rho}_{\rm css}^{(H)}$ is given by ($q = p/2$):

$$\hat{\rho}_{\rm css}^{(H)}(p) = q(1-q)\sum_{j,k=0}^{1}(-1)^{j-k}|j,1-j\rangle\langle k,1-k| + (1-q)^2|00\rangle\langle 00| + q^2|11\rangle\langle 11|. \tag{29}$$

By applying Vedral-Plenio's theorem [15], the REE can be found as follows [25]:

$$E_R(\hat{\rho}^{(H')}) \equiv E_R^{(H')}(p,N) = q^2 x \lg x + 2qy_1 \lg\left(\frac{y_1}{1-q}\right) + y_2 \lg\left(\frac{y_2}{(1-q)^2}\right), \tag{30}$$

where $y_1 = 1 - qx$ and $y_2 = 1 - 2q + q^2x$. With this choice of x, parameter N is just the negativity of $\hat{\rho}^{(H')}(p,N)$. States corresponding to blue region in Fig. 4 can be obtained as special cases of state $\hat{\rho}^{(H')}(p,N)$ for N in the range $0 < N < N_Y$ and proper values of p. Thus, it is seen that there are mixed states for which the REE is greater than that for pure states at least in the range $N \in (0, N_Y)$. Later, in Ref. [28], it was shown that the generalized Horodecki states exhibit this property in slightly larger range as shown by yellow region in Fig. 4. There is some evidence [28] that the upper bound of the REE as a function of the negativity is likely to be given by these states.

Recently, we also analytically demonstrated [30] that the entanglement REE for a given nonlocality for mixed states exceeds that for pure states [see Fig. 3(f)]. Moreover, this effect occurs in the larger range of abscissa values in comparison to the dependence of the REE on the negativity, as seen by comparing Figs 3(e) and 3(f).

6. Conclusion

In this short review, we presented a few intriguing properties of some standard entanglement measures for two qubits. Our examples include a comparison of the negativity corresponding to the Peres-Horodecki criterion [12,13], the Wootters concurrence [9], and the relative entropy of entanglement of Vedral et al. [14]. Moreover, the predictions of these measures were also compared with the Horodecki measure [17] of the violation of Bell's inequality, referred here to as "nonlocality".

We discussed the following three counterintuitive properties of entanglement measures: (i) entangled states *cannot* be ordered uniquely with the entanglement measures, which also implies that (ii) fragility or robustness of entanglement of dissipative systems *cannot* be uniquely classified by entanglement measures, and (iii) there are two-qubit mixed states, which are more entangled (according to the REE) than pure states for a given negativity or nonlocality.

It is well known that there is no unique entanglement measure for mixed states. But the relativity of entanglement measures and its implications are more counterintuitive. Our demonstration might indicate that operational approaches to the quantum entanglement problem are more meaningful rather than standard approaches based on formally-defined measures. We find the problem of defining operational entanglement measures analogous to operational approaches to the quantum phase problem [1] posed by Noh et al. [34,35]. The idea is to define entanglement (or phase) measures in terms of what actually is, or can be, measured.

We hope that the discussed problem of non-unique ordering of states according to formally-defined entanglement measures can stimulate investigations of operationally-defined measures oriented for some specific experiments.

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Appendix A: Notes on the calculation of the REE

The concurrence, negativity and nonlocality can be calculated easily. By contrast, there has not yet been proposed an efficient method to calculate the REE for arbitrary mixed states even in the case of two qubits [36]. Analytical formulas for the REE are known only for some special sets of states with high symmetry (see [5,37] and references therein). Thus, usually, numerical methods for calculating the REE have to be applied [15,38,39].

It is a long-standing problem, posed by Eisert [36], of obtaining an analytical compact formula for the REE for two qubits. The problem is equivalent to finding the closest separable state $\hat{\rho}_{\rm css}$ for a given entangled state $\hat{\rho}$. In Ref. [37], a few arguments were given indicating that this problem, probably, cannot

[1] Noh et al. in Ref. [35] wrote: "There has been a good deal of discussion in the past of the most appropriate dynamical variable to represent the phase of a quantum field, and many candidates have been studied. Our analysis suggests that this question may not have a general answer with respect to the measured phase operators, because different measurement schemes lead to different operators. As in many other quantum-mechanical problems, it seems that questions about the value of a dynamical variable cannot be divorced from the measurement process that generates the ensemble."

be solved analytically for arbitrary states. Nevertheless, there exists a solution to the inverse problem of finding an analytical formula for $\hat{\rho}$ for a given closest separable state $\hat{\rho}_{\rm css}$ as derived by Ishizaka et al. [40,37].

The complexity of the problem can be explained (see, e.g., Ref. [15]) by virtue of Caratheodory's theorem, which implies that any separable two-qubit state can be decomposed as

$$\hat{\rho}_{\rm sep} = \sum_{j=1}^{16} p_j^2 |\psi_j^{(1)}\rangle\langle\psi_j^{(1)}| \otimes |\psi_j^{(2)}\rangle\langle\psi_j^{(2)}|, \tag{A1}$$

where the kth ($k = 1, 2$) qubit pure states can be parametrized, e.g., as follows

$$|\psi_j^{(k)}\rangle = \cos\alpha_j^{(k)}|0\rangle + \exp(i\eta_j^{(k)})\cos\alpha_j^{(k)}|1\rangle, \tag{A2}$$

and $p_j = \sin\phi_{j-1}\prod_{i=j}^{15}\cos\phi_i$ with $\phi_0 = \pi/2$. Thus, the minimalization of the quantum relative entropy $S(\hat{\rho}||\hat{\rho}_{\rm sep})$, given by Eq. (5), with $\hat{\rho}_{\rm sep}$ described by Eq. (A1), should be performed over $16 \times 4 + 15 = 79$ real parameters. Usually (see, e.g., Refs. [15,38]), gradient-type algorithms are applied to perform the minimalization. Řeháček and Hradil [38] proposed a method resembling a state reconstruction based on the maximum likelihood principle. Doherty et al. [39] designed a hierarchy of more and more complex operational separability criteria for which convex optimization methods (known as semidefinite programs) can be applied efficiently. One can also use an iterative method based on Ishizaka formula [40, 37] for the closest entangled state for a given separable state in order to find the closest separable state for a given entangled state. Our algorithms for calculating the REE are based either on the latter method or on a simplex search method without using numerical or analytic gradients.

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