Sudden vanishing of spin squeezing under decoherence

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In order to witness multipartite correlations beyond pairwise entanglement, spin-squeezing parameters are analytically calculated for a spin ensemble in a collective initial state under three different decoherence channels. It is shown that, in analogy to pairwise entanglement, the spin squeezing described by different parameters can suddenly become zero at different vanishing times. This finding shows the general occurrence of sudden vanishing phenomena of quantum correlations in many-body systems, which here is referred to as spin-squeezing sudden death (SSSD). It is shown that the SSSD usually occurs due to decoherence and that SSSD never occurs for some initial states in the amplitude-damping channel. We also analytically obtain the vanishing times of spin squeezing.

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I. INTRODUCTION

Quantum entanglement [1] plays an important role in both the foundations of quantum physics and quantum-information processing [2]. Moreover, various entangled states have been produced in many experiments for different goals when studying various nonclassical phenomena and their applications [3–11]. Thus, entanglement is a quantum resource, and how to measure and detect entanglement is very crucial for both theoretical investigations and potential practical applications.

For a system of two spin-$1/2$ particles or a composite system of a spin-$1/2$ and a spin-1, there are operationally computable entanglement measures such as concurrence [12] and negativity [13,14], but no universal measures have been found for general many-body systems. To overcome this difficulty, entanglement witnesses are presented to detect some kinds of entanglement in many-body systems [14,15]. Now it is believed that spin squeezing [16,17] may be useful for this task [18–20]. In a general sense, spin-squeezing parameters are multipartite entanglement witnesses. For a class of many-particle states, it has been proved that the concurrence is linearly related to some squeezing parameters [21]. In fact, spin-squeezing parameters [16–19] could be calculated also in a simple operational fashion, which characterizes multipartite quantum correlations beyond the pairwise entanglement. Another important reason for choosing spin-squeezing parameters as indicators of multipartite correlations is that spin squeezing is relatively easy to generate [17,22] and measure experimentally [23,24].

Besides being a parameter characterizing multipartite correlations, spin squeezing is physically natural for controlling many-body systems. It is difficult to control a quantum many-body system since its constituents cannot be individually addressed. In this sense, one needs to use collective operations, and spin squeezing is one of the most successful approaches for controlling such systems. For example, creating spin squeezing of an atomic ensemble could result in precision measurements based on many-atom spectroscopy [17]. Therefore, we can also regard spin squeezing as a quantum resource since for more than two particles it behaves as two-particle entanglement in controlling and detecting quantum correlations. On this quantum resource, we need to further consider the effects of decoherence [25,26]. Thus, it is important to study the environment-induced decoherence effects on both spin squeezing and multipartite entanglement [27–37]. A decaying time evolution of the spin squeezing under decoherence [27,38–40] can be used to analyze whether this quantum resource is robust.

In this article we address this problem by calculating three spin-squeezing parameters for a spin ensemble in a collective excited state. We study the time evolution of spin squeezing under local decoherence, acting independently and equally on each spin. Here, the irreversible processes are modelled as three decoherence channels: the amplitude damping, pure dephasing, and depolarizing channels. We find that, similar to the sudden death of pairwise entanglement [41], spin squeezing can also suddenly vanish with different lifetimes for some decoherence channels, showing in general different vanishing times in multipartite correlations in quantum many-body systems. Thus, similar to the discovery of pairwise entanglement sudden death (ESD) [41], the spin-squeezing sudden death (SSSD) occurs due to decoherence. We will see that for some initial states, the SSSD never occurs under the amplitude-damping channel. We also give analytical expressions for the vanishing time of spin squeezing and pairwise entanglement. The ESD has been tested experimentally [39,42] and we also expect that the SSSD can also be realized experimentally.

This article is organized as follows. In Sec. II, we introduce the initial state from the one-axis twisting Hamiltonian and then, in Sec. III, the decoherence channels. In Sec. IV, we list three parameters of spin squeezing and discuss the relations among them. For a necessary comparison, the concurrence is also calculated. We also study initial-state squeezing. In Sec. V, we study three different types of spin squeezing and concurrence under three different decoherence channels. Both analytical and numerical results are given. We conclude in Sec. VI.

II. INITIAL STATE

We consider an ensemble of N spin-1/2 particles with ground state $|1\rangle$ and excited state $|0\rangle$. This system has exchange symmetry, and its dynamical properties can be described by the collective operators

$$J_\alpha = \sum_{k=1}^{N} j_{k\alpha} = \frac{1}{2}\sum_{k=1}^{N}\sigma_{k\alpha} \tag{1}$$

for $\alpha = x, y, z$. Here, $\sigma_{k\alpha}$ are the Pauli matrices for the kth qubit. To study the decoherence of spin squeezing, we choose a state which is initially squeezed. One typical class of such spin-squeezed states is the one-axis twisting collective spin state [16],

$$|\Psi(\theta_0)\rangle_0 = e^{-i\theta_0 J_x^2/2}|1\rangle^{\otimes N} = e^{-i\theta_0 J_x^2/2}|\mathbf{1}\rangle, \tag{2}$$

which could be prepared by the one-axis twisting Hamiltonian

$$H = \chi J_x^2, \tag{3}$$

where

$$\theta_0 = 2\chi t \tag{4}$$

is the *one-axis twist angle* and χ is the coupling constant. For this state, it was proved [21] that the spin squeezing ξ_1^2 [16] and the concurrence C_0 [12] are equivalent since there exists a linear relation

$$\xi_1^2 = 1 - (N-1)C_0$$

between them. Physically, they occur and disappear simultaneously. The spin squeezing of this state can be generated and stored in, e.g., a two-component Bose-Einstein condensate [43].

A. Initial-state symmetry

The initial state has an obvious symmetry resulting from Eq. (2), the so-called even-parity symmetry, which means that only even excitations of spins occur in the state. Since J_α define an angular-momentum spinor representation of SO(3), the general definitions of spin squeezing for abstract operators J_x, J_y, and J_z can work well by identifying $N/2$ with the highest weight J, which corresponds to the collective ground state

$$|J, -J\rangle = |1\rangle^{\otimes N} \equiv |\mathbf{1}\rangle, \tag{5}$$

indicating that all spins are in the ground state. The symmetric space is generated by the collective operator

$$J_+ = \frac{1}{2}\sum_{k=1}^{N}\sigma_{k+}$$

acting on the collective ground state. Here,

$$\sigma_{k\pm} = \tfrac{1}{2}(\sigma_{kx} \pm i\sigma_{ky}).$$

In others words, the state is in the maximally symmetric space spanned by the Dicke states. So, the N spin-1/2 system behaves like a larger spin-$N/2$ system. It can be proved that any pure state with exchange symmetry belongs to the above-mentioned symmetric space, but for mixed states the state space can be extended to include a space beyond the symmetric one [44]. In the following discussions, we focus on such an extended space.

In fact, after decoherence, not only the symmetric Dicke states are populated, but also states with lower symmetry. So, it is not sufficient to describe the system in only $(N+1)$-dimensional space. Although the maximal symmetry is broken, the exchange symmetry is not affected by the decoherence as each local decoherence equally acts on each spin. In other words, a state with exchange symmetry does not necessarily belong to the maximally symmetric space.

With only the exchange symmetry, from Eq. (1), the global expectations or correlations of collective operators are obtained as

$$\langle J_\alpha^2\rangle = \frac{N}{4} + \frac{N(N-1)}{4}\langle\sigma_{1\alpha}\sigma_{2\alpha}\rangle, \tag{6}$$

$$\langle J_-^2\rangle = N(N-1)\langle\sigma_{1-}\sigma_{2-}\rangle, \tag{7}$$

$$\langle [J_x, J_y]_+\rangle = \frac{N(N-1)}{4}\langle[\sigma_{1x}, \sigma_{2y}]_+\rangle. \tag{8}$$

Furthermore, it follows from Eq. (6) that

$$\langle J_x^2 + J_y^2\rangle = \frac{N}{2} + \frac{N(N-1)}{2}\langle\sigma_{1+}\sigma_{2-} + \sigma_{1-}\sigma_{2+}\rangle, \tag{9}$$

$$\langle J_x^2 + J_y^2 + J_z^2\rangle = \frac{N^2}{4}\left[\frac{3}{N} + \left(1 - \frac{1}{N}\right)\langle\vec{\sigma}_1\cdot\vec{\sigma}_2\rangle\right]. \tag{10}$$

These equations show the relations between the global and local expectations and correlations, which are useful in the following calculations.

III. DECOHERENCE CHANNELS AND EXAMPLES OF THEIR IMPLEMENTATIONS

Having introduced the initial state, now we discuss three typical decoherence channels: the amplitude-damping channel (ADC), the phase-damping channel (PDC), and the depolarizing channel (DPC).

These channels are prototype models of dissipation relevant in various experimental systems. They provide "a revealing caricature of decoherence in realistic physical situations, with all inessential mathematical details stripped away" [45]. But yet this "caricature of decoherence" leads to theoretical predictions being often in good agreement with experimental data. Examples include multiphoton systems, ion traps, atomic ensembles, or a solid-state spin systems such as quantum dots or NV diamonds, where qubits are encoded in electron or nuclear spins.

Here, we briefly describe only a few of such implementations.

A. Amplitude-damping channel

The ADC is defined as

$$\mathcal{E}_{\rm ADC}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger, \tag{11}$$

where

$$E_0 = \sqrt{s}|0\rangle\langle 0| + |1\rangle\langle 1|, \quad E_1 = \sqrt{p}|1\rangle\langle 0| \tag{12}$$

are the Kraus operators, $p = 1 - s$, $s = \exp(-\gamma t/2)$, and γ is the damping rate. In the Bloch representation, the ADC squeezes the Bloch sphere into an ellipsoid and shifts it toward the north pole. The radius in the xy plane is reduced by a factor $\sqrt{s}$, while in the z direction it is reduced by a factor s.

The ADC is a prototype model of a dissipative interaction between a qubit and its environment. For example, the ADC model can be applied to describe the spontaneous emission of a photon by a two-level system into an environment of photon or phonon modes at zero (or very low) temperature in (usually) the weak Born-Markov approximation. The ADC can also describe processes contributing to T_1-relaxation in spin resonance at zero temperature. Note that by introducing an "upward" decay (i.e., a decay toward the south pole of the Bloch sphere), in addition to the standard "downward" decay, the ADC can be used to describe dissipation into the environment also at finite temperature.

The ADC acting on a system qubit in an unknown state ρ can be implemented in a two-qubit circuit performing a rotation $R_y(\theta)$ of an ancilla qubit (initially in the ground state) controlled by the system qubit and followed by a controlled-NOT (CNOT) gate on the system qubit controlled by the ancilla qubit [2]. The parameter θ is simply related to the probability p in Eq. (11). The ancilla qubit, which models the environment, is measured after the gate operation.

The ADC-induced sudden vanishing of entanglement was first experimentally demonstrated for polarization-encoded qubits [42]. For this reason let us shortly describe this optical implementation of the ADC. It is based on a Sagnac-type ring interferometer composed of a polarizing beam splitter and a half-wave plate at an angle corresponding to the parameter p in Eq. (11). The beam splitter separates an incident beam (being in a superposition of states with horizontal, $|H\rangle$, and vertical, $|V\rangle$, polarizations) into spatially distinct counterpropagating light beams. The H component leaves the interferometer unchanged. But the V component is rotated in the wave plate, which corresponds to probabilistic damping into the H component. Then, at the exit from the interferometer, this component is probabilistically transmitted or reflected from the beam splitter. So it is cast into two orthogonal spatial modes corresponding the reservoir states with and without excitation.

The action of the ADC can be represented by an interaction Hamiltonian [2]: $H \sim ab^\dagger + a^\dagger b$, where a ($a^\dagger$) and b ($b^\dagger$) are annihilation (creation) operators of the system and environment oscillators, respectively. In more general models of damping, a single oscillator b of the reservoir is replaced by a finite or infinite collection of oscillators $\{b_n\}$ coupled to the system oscillator with different strengths (see, e.g., Refs. [46,47]). For the example of quantum states of motion of ions trapped in a radiofrequency (Paul) trap, the amplitude damping can be modeled by coupling an ion to the motional amplitude reservoir described by the above multioscillator Hamiltonian [47]. The high-temperature reservoir is possible to simulate by applying (on trap electrodes) a random uniform electric field with spectral amplitude at the ion motional frequency [48,49]. The zero-temperature reservoir can be simulated by laser cooling combined with spontaneous Raman scattering [50].

B. Phase-damping channel

The PDC is a prototype model of dephasing or pure decoherence, i.e., loss of coherence of a two-level state without any loss of system's energy. The PDC is described by the map

$$\mathcal{E}_{\mathrm{PDC}}(\rho) = s\rho + p\,(\rho_{00}|0\rangle\langle 0| + \rho_{11}|1\rangle\langle 1|)\,, \qquad (13)$$

and obviously the three Kraus operators are given by

$$E_0 = \sqrt{s}\mathbb{1}, \quad E_1 = \sqrt{p}|0\rangle\langle 0|, \quad E_2 = \sqrt{p}|1\rangle\langle 1|, \quad (14)$$

where $\mathbb{1}$ is the identity operator. For the PDC, there is no energy change and a loss of decoherence occurs with probability p. As a result of the action of the PDC, the Bloch sphere is compressed by a factor $(1 - 2p)$ in the xy plane.

In analogy to the ADC, the PDC can be considered as an interaction between two oscillators (modes) representing system and environment as described by the interaction Hamiltonian: $H \sim a^\dagger a(b^\dagger + b)$ [2]. In more general phase-damping models, a single environmental mode b is usually replaced by an infinite collection of modes b_n coupled, with various strengths, to mode a.

It is evident that the action of the PDC is nondissipative. It means that, in the standard computational basis $|0\rangle$ and $|1\rangle$, the diagonal elements of the density matrix ρ remain unchanged, while the off-diagonal elements are suppressed. Moreover, the qubit states $|0\rangle$ and $|1\rangle$ are also unchanged under the action of the PDC, although any superposition of them (i.e., any point in the Bloch sphere, except the poles) becomes entangled with the environment.

The PDC can be interpreted as elastic scattering between a (two-level) system and a reservoir. It is also a model of coupling a system with a noisy environment via a quantum nondemolition (QND) interaction. Note that spin squeezing of atomic ensembles can be generated via QND measurements [10,24,51–55]. So modeling the spin-squeezing decoherence via the PDC can be relevant in this context. The PDC is also a suitable model to describe T_2 relaxation in spin resonance. This in contrast to modeling T_1 relaxation via the ADC.

A circuit modeling the PDC can be realized as a simplified version of the circuit for the ADC, discussed in the previous subsection, obtained by removing the CNOT gate [2]. Then, the angle θ in the controlled rotation gate $R_y(\theta)$ is related to the probability p in Eq. (13).

The sudden vanishing of entanglement under the PDC was first experimentally observed in Ref. [42]. This optical implementation of the PDC was based on the same system as the above-mentioned Sagnac interferometer for the ADC but with an additional half-wave plate at a $\pi/4$ angle in one of the outgoing modes.

Some specific kinds of PDCs can be realized in a more straightforward manner. For example, in experiments with trapped ions, the motional PDC can be implemented just by modulating the trap frequency, which changes the phase of the harmonic motion of ions [48,49] (for a review see Ref. [47] and references therein).

C. Depolarizing channel

The definition of the DPC is given via the map

$$\mathcal{E}_{\rm DPC}(\rho) = \sum_{i=0}^{3} E_k \rho E_k^{\dagger}, \\ = (1-p')\rho + \frac{p'}{3}(\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z), \quad (15)$$

where

$$E_0 = \sqrt{1-p'}\mathbb{1}, \quad E_1 = \sqrt{\frac{p'}{3}}\sigma_x, \\ E_2 = \sqrt{\frac{p'}{3}}\sigma_y, \quad E_3 = \sqrt{\frac{p'}{3}}\sigma_z, \quad (16)$$

are the Kraus operators. By using the following identity

$$\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z + \rho = 2\mathbb{1},$$

we obtain

$$\mathcal{E}_{\rm DPC}(\rho) = s\rho + p\frac{\mathbb{1}}{2}, \quad (17)$$

where $p = 4p'/3$. We see that for the DPC, the spin is unchanged with probability $s = 1 - p$ or it is depolarized to the maximally mixed state $\mathbb{1}/2$with probability p. It is seen that due to the action of the DPC, the radius of the Bloch sphere is reduced by a factor s, but its shape remains unchanged.

Formally, the action of the DPC on a qubit in an unknown state ρ can be implemented in a three-qubit circuit composed of two CNOT gates with two auxiliary qubits initially in mixed states

$$\rho_1 = \mathbb{1}/2, \quad \rho_2 = (1-p)|00\rangle\langle 00| + p|11\rangle\langle 11|, \quad (18)$$

which model the environment. Qubit ρ_2 controls the other qubits via the CNOT gates [2].

The DPC map can also be implemented by applying each of the Pauli operators $[\mathbb{1}, \sigma_x, \sigma_y, \sigma_z]$ at random with the same probability. Using this approach, optical DPCs have been realized experimentally both in free space [56] and in fibers [57], where qubits are associated with polarization states of single photons. In Ref. [56], the DPC was implemented by using a pair of equal electro-optical Pockels cells. One of them was performing a σ_x gate and the other a σ_y gate. The simultaneous action of both σ_x and σ_y corresponds to a σ_y gate. The cells were driven (with a mutual delay of $\tau/2$) by a continuous-wave periodic square-wave electric field with a variable pulse duration τ, so the total depolarizing process lasted 2τ for each period.

Analogous procedures can be implemented in other systems, including collective spin states of atomic ensembles. The coherent manipulation of atomic spin states by applying off-resonantly coherent pulses of light is a basic operation used in many applications [58]. We must admit that the standard methods enable rotations in the Bloch sphere of only classical spin states (i.e., coherent spin states). Nevertheless, recently [24] an experimental method has been developed to rotate also spin-squeezed states.

It is worth noting that in experimental realizations of decoherence channels (e.g., in ion-trap systems [59]), sufficient resources for complete quantum tomography are provided even for imperfect preparation of input states and the imperfect measurements of output states from the channels.

IV. SPIN-SQUEEZING DEFINITIONS AND CONCURRENCE

Now, we discuss several parameters of spin squeezing and give several relations among them. To compare spin squeezing with pairwise entanglement, we also give the definition of concurrence. We notice that most previous investigations on ESD of concurrence were only carried out for two-particle system rather than for two-particle subsystem embedded in a larger system. For the initial states, spin-squeezing parameters and concurrence are also given below.

A. Spin-squeezing parameters and their relations

1. Definitions of spin squeezing

There are several spin-squeezing parameters, but we list only three typical and related ones as follows [16–19]:

$$\xi_1^2 = \frac{4(\Delta J_{\vec{n}_\perp})^2_{\rm min}}{N}, \quad (19)$$

$$\xi_2^2 = \frac{N^2}{4\langle\vec{J}\rangle^2}\xi_1^2, \quad (20)$$

$$\xi_3^2 = \frac{\lambda_{\rm min}}{\langle\vec{J}^2\rangle - \frac{N}{2}}. \quad (21)$$

Here, the minimization in the first equation is over all directions denoted by $\vec{n}_\perp$, perpendicular to the mean spin direction $\langle\vec{J}\rangle/\langle\vec{J}^2\rangle$; $\lambda_{\rm min}$ is the minimum eigenvalue of the matrix [19]

$$\Gamma = (N-1)\gamma + \mathbf{C}, \quad (22)$$

where

$$\gamma_{kl} = C_{kl} - \langle J_k\rangle\langle J_l\rangle \quad \text{for} \quad k,l \in \{x,y,z\} = \{1,2,3\}, \quad (23)$$

is the covariance matrix and $\mathbf{C} = [C_{kl}]$ with

$$C_{kl} = \tfrac{1}{2}\langle J_l J_k + J_k J_l\rangle \quad (24)$$

is the global correlation matrix. The parameters ξ_1^2, ξ_2^2, and ξ_3^2 were defined by Kitagawa and Ueda [16], Wineland *et al.* [17], and Tóth *et al.* [19], respectively. If $\xi_2^2 < 1$ ($\xi_3^2 < 1$), spin squeezing occurs, and we can safely say that the multipartite state is entangled [18,19]. Although we cannot say that the squeezed state via the parameter ξ_1^2 is entangled, it is indeed closely related to quantum entanglement [21].

2. Squeezing parameters for states with parity

We know from Sec. II A that the initial state has an even parity and that the mean spin direction is along the z direction. During the transmission through all the three decoherence channels discussed here, the mean spin direction does not change. For states with a well-defined parity (even or odd), the spin-squeezing parameter ξ_1^2 was found to be [21]

$$\xi_1^2 = \frac{2}{N}\left(\langle J_x^2 + J_y^2\rangle - |\langle J_-^2\rangle|\right). \quad (25)$$

Then, the parameter ξ_2^2 given by Eq. (20) becomes

$$\xi_2^2 = \frac{N^2\xi_1^2}{4\langle J_z\rangle^2} = \frac{N\left(\langle J_x^2 + J_y^2\rangle - |\langle J_-^2\rangle|\right)}{2\langle J_z\rangle^2}. \tag{26}$$

For the third squeezing parameter (see Appendix A for the derivation), we have

$$\xi_3^2 = \frac{\min\left\{\xi_1^2, \varsigma^2\right\}}{4N^{-2}\langle \vec{J}^2\rangle - 2N^{-1}}, \tag{27}$$

where

$$\varsigma^2 = \frac{4}{N^2}[N(\Delta J_z)^2 + \langle J_z\rangle^2]. \tag{28}$$

Note that the first parameter ξ_1^2 becomes a key ingredient for the latter two squeezing parameters (ξ_2^2 and ξ_3^2).

3. Spin-squeezing parameters in terms of local expectations

For later applications, we now express the squeezing parameters in terms of local expectations and correlations, and also examine the meaning of ς^2, which will be clear by substituting Eqs. (1) and (6) into Eq. (28),

$$\begin{aligned}\varsigma^2 &= 1 + \mathcal{C}_{zz}\\ &= 1 + (N-1)\left(\langle\sigma_{1z}\sigma_{2z}\rangle - \langle\sigma_{1z}\rangle\langle\sigma_{2z}\rangle\right).\end{aligned} \tag{29}$$

Thus, the parameter ς^2 is simply related to the correlation $\mathcal{C}_{zz}$ along the z direction. A negative correlation $\mathcal{C}_{zz} < 0$ is equivalent to $\varsigma^2 < 1$. It is already known that the spin squeezing parameter ξ_1^2 can be written as [60]

$$\xi_1^2 = 1 + (N-1)\mathcal{C}_{\vec{n}_\perp\vec{n}_\perp}, \tag{30}$$

where $\mathcal{C}_{\vec{n}_\perp\vec{n}_\perp}$ is the correlation function in the direction perpendicular to the mean spin direction. So, the spin squeezing $\xi_1^2 < 1$ is equivalent to the negative pairwise correlations $\mathcal{C}_{\vec{n}_\perp\vec{n}_\perp} < 0$ [60].

Thus, from the above analysis, spin squeezing and negative correlations are closely connected to each other. The parameter $\varsigma^2 < 1$ indicates that spin squeezing occurs along the z direction, and $\xi_1^2 < 1$ implies spin squeezing along the direction perpendicular to the mean spin direction. Furthermore, from Eq. (27), a competition between the transverse and longitudinal correlations is evident.

By substituting Eqs. (7) and (9) to Eq. (25), one can obtain the expression of ξ_1^2 in terms of local correlations $\langle\sigma_{1+}\sigma_{2-}\rangle$ and $\langle\sigma_{1-}\sigma_{2-}\rangle$ as follows:

$$\begin{aligned}\xi_1^2 &= 1 + (N-1)\langle\sigma_{1+}\sigma_{2-} + \sigma_{1-}\sigma_{2+}\rangle - 2(N-1)|\langle\sigma_{1-}\sigma_{2-}\rangle|\\ &= 1 + 2(N-1)\langle\sigma_{1+}\sigma_{2-}\rangle - |\langle\sigma_{1-}\sigma_{2-}\rangle|).\end{aligned} \tag{31}$$

The second equality in Eq. (31) results from the exchange symmetry. From Eqs. (1), (10), and (29), one finds

$$\xi_2^2 = \frac{\xi_1^2}{\langle\sigma_{1z}\rangle^2}, \tag{32}$$

$$\xi_3^2 = \frac{\min\left\{\xi_1^2, 1 + \mathcal{C}_{zz}\right\}}{(1-N^{-1})\langle\vec{\sigma}_1\cdot\vec{\sigma}_2\rangle + N^{-1}}. \tag{33}$$

Thus, we have reexpressed the squeezing parameters in terms of local correlations and expectations.

4. New spin-squeezing parameters

In order to characterize spin squeezing more conveniently, we define the following squeezing parameters:

$$\zeta_k^2 = \max\left(0, 1 - \xi_k^2\right), \quad k \in \{1, 2, 3\}. \tag{34}$$

This definition is similar to the expression of the concurrence given below. Spin squeezing appears when $\zeta_k^2 > 0$, and there is no squeezing when ζ_k^2 vanishes. Thus, the definition of the first parameter ζ_1^2 has a clear meaning, namely, it is the *strength* of the negative correlations as seen from Eq. (30). The larger is ζ_1^2, the larger is the strength of the negative correlation, and the larger of is the squeezing. More explicitly, for the initial state, we have $\xi_1^2 = 1 - (N-1)C_0$ [21], so ζ_1^2 is just the rescaled concurrence $\zeta_1^2 = C_r(0) = (N-1)C_0$ [61].

Here, we give a few comments on the spin-squeezing parameter ξ_2^2, which represents a competition between ξ_1^2 and $\langle\sigma_{1z}\rangle^2$: the state is squeezed according to the definition of ξ_2^2 if $\xi_1^2 < \langle\sigma_{1z}\rangle^2$. We further note that [62]

$$\langle\sigma_{1z}\rangle^2 = 1 - 2E_L, \tag{35}$$

where E_L is the linear entropy of one spin and it can be used to quantify the entanglement of pure states [14]. So, there is a competition between the strength of negative correlations and the linear entropy $2E_L$ in the parameter ξ_2^2, and $\zeta_1^2 > 2E_L$ implies the appearance of squeezing.

B. Concurrence for pairwise entanglement

It has been found that the concurrence is closely related to spin squeezing [21]. Here, we consider its behavior under various decoherence channels. The concurrence quantifying the entanglement of a pair of spin-1/2 can be calculated from the reduced density matrix. It is defined as [12]

$$C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{36}$$

where the quantities λ_i are the square roots of the eigenvalues in descending order of the matrix product

$$\varrho_{12} = \rho_{12}(\sigma_{1y}\otimes\sigma_{2y})\rho_{12}^*(\sigma_{1y}\otimes\sigma_{2y}). \tag{37}$$

In (37), ρ_{12}^* denotes the complex conjugate of ρ_{12}.

The two-spin reduced density matrix for a parity state with the exchange symmetry can be written in a block-diagonal form [63]

$$\rho_{12} = \begin{pmatrix} v_+ & u^* \\ u & v_- \end{pmatrix} \oplus \begin{pmatrix} w & y \\ y & w \end{pmatrix}, \tag{38}$$

in the basis $\{|00\rangle, |11\rangle, |01\rangle, |10\rangle\}$, where

$$v_\pm = \tfrac{1}{4}\left(1 \pm 2\langle\sigma_{1z}\rangle + \langle\sigma_{1z}\sigma_{2z}\rangle\right), \tag{39}$$

$$w = \tfrac{1}{4}\left(1 - \langle\sigma_{1z}\sigma_{2z}\rangle\right), \tag{40}$$

$$u = \langle\sigma_{1+}\sigma_{2+}\rangle, \tag{41}$$

$$y = \langle\sigma_{1+}\sigma_{2-}\rangle. \tag{42}$$

The concurrence is then given by [64]

$$C = \max\{0, 2(|u| - w), 2(y - \sqrt{v_+v_-})\}. \tag{43}$$

From the above expressions of the spin-squeezing parameters and concurrence, we notice that if we know the expectation $\langle\sigma_{1z}\rangle$, and the correlations $\langle\sigma_{1+}\sigma_{2-}\rangle$, $\langle\sigma_{1-}\sigma_{2-}\rangle$, and $\langle\sigma_{1z}\sigma_{2z}\rangle$, all the squeezing parameters and concurrence can be determined. Below, we will give explicit analytical expressions for them subject to three decoherence channels.

C. Initial-state squeezing and concurrence

We will now investigate initial spin squeezing and pairwise entanglement by using our results for the spin-squeezing parameters and concurrence obtained in the last subsections. We find that the third squeezing parameter ξ_3^2 is equal to the first one ξ_1^2. The squeezing parameter ξ_1^2 is given by (see Appendix B):

$$\begin{aligned}\xi_1^2(0) &= 1 - C_r(0)\\ &= 1-(N-1)C_0,\\ &= 1-2(N-1)(|u_0|-y_0),\end{aligned} \tag{44}$$

where

$$C_0 = \tfrac{1}{4}\{[(1-\cos^{N-2}\theta_0)^2 + 16\sin^2(\theta_0/2)\cos^{2N-4}(\theta_0/2)]^{\frac{1}{2}} - 1 + \cos^{N-2}\theta_0\} \tag{45}$$

is the concurrence [21].

The parameter $\xi_2^2(0)$ is easily obtained, as we know both $\xi_1^2(0)$ and $\langle\sigma_{1z}\rangle_0^2$ (B6). For this state, following from Eq. (10), $\langle\vec{\sigma}_1\cdot\vec{\sigma}_2\rangle_0 = 1$, and thus the third parameter given by Eq. (33) becomes

$$\xi_3^2(0) = \min\left[\xi_1^2(0), \varsigma^2(0)\right] = \min[\{1 - C_r(0), 1 + \mathcal{C}_{zz}(0)], \tag{46}$$

where the correlation function is

$$\mathcal{C}_{zz}(0) = \tfrac{1}{2}(1+\cos^{N-2}\theta_0) - \cos^{2N-2}(\theta_0/2) \geqslant 0. \tag{47}$$

The proof of the above inequality is given in Appendix C.

As the correlation function $\mathcal{C}_{zz}(0)$ and the concurrence $C_r(0)$ are always $\geqslant 0$, Eq. (46) reduces to

$$\xi_3^2(0) = \xi_1^2(0) = 1 - C_r(0). \tag{48}$$

So, for the initial state, the spin-squeezing parameters $\xi_3^2(0)$ and $\xi_1^2(0)$ are equal or equivalently, we can write $\zeta_1^2(0) = \zeta_3^2(0) = C_r(0)$ according to the definition of parameter ζ_k^2 given by Eq. (34). Below we made a summary of results of this section in Table I.

V. SPIN SQUEEZING UNDER DECOHERENCE

Now we begin to study spin squeezing under three different decoherence channels. From the previous analysis, all the spin-squeezing parameters and the concurrence are determined by some correlation functions and expectations. So, if we know the evolution of them under decoherence, the evolution of any squeezing parameters and pairwise entanglement can be calculated.

A. Heisenberg approach

We now use the Heisenberg picture to calculate the correlation functions and the relevant expectations. A decoherence channel with Kraus operators K_μ is defined via the map

$$\mathcal{E}(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger. \tag{49}$$

Then, an expectation value of the operator A can be calculated as $\langle A\rangle = \mathrm{Tr}[A\mathcal{E}(\rho)]$. Alternatively, we can define the following map,

$$\mathcal{E}^\dagger(\rho) = \sum_\mu K_\mu^\dagger \rho K_\mu. \tag{50}$$

It is easy to check that

$$\langle A\rangle = \mathrm{Tr}[A\mathcal{E}(\rho)] = \mathrm{Tr}[\mathcal{E}^\dagger(A)\rho]. \tag{51}$$

So, one can calculate the expectation value via the above equation (51). This is very similar to the standard Heisenberg picture.

B. Amplitude-damping channel

1. Squeezing parameters

Based on the above approach and the Kraus operators for the ADC given by Eq. (12), we now find the evolutions of the following expectations under decoherence (see Appendix D for details)

$$\langle\sigma_{1z}\rangle = s\langle\sigma_{1z}\rangle_0 - p, \tag{52a}$$

$$\langle\sigma_{1-}\sigma_{2-}\rangle = s\langle\sigma_{1-}\sigma_{2-}\rangle_0, \tag{52b}$$

$$\langle\sigma_{1+}\sigma_{2-}\rangle = s\langle\sigma_{1+}\sigma_{2-}\rangle_0, \tag{52c}$$

$$\langle\sigma_{1z}\sigma_{2z}\rangle = s^2\langle\sigma_{1z}\sigma_{2z}\rangle_0 - 2sp\langle\sigma_{1z}\rangle_0 + p^2. \tag{52d}$$

TABLE I. Spin-squeezing parameters ξ_1^2 [16], ξ_2^2 [17], ξ_3^2 [19] and concurrence C [12] for arbitrary states (first two columns), states with parity (third column). The squeezing parameters are also expressed in terms of local expectations (fourth column) and in terms of the initial rescaled concurrence $C_r(0)$ for initial states (last column). Also, C_0 is the initial concurrence, and other parameters are defined in the text.

Squeezing parameters	Definitions	States with parity	In terms of local expectations	Initial state
ξ_1^2	$\frac{4(\Delta J_{\vec{n}_\perp})^2_{\min}}{N}$	$\frac{2}{N}\left(\langle J_x^2+J_y^2\rangle - \lvert\langle J_-^2\rangle\rvert\right)$	$1+2(N-1)(y-\lvert u\rvert)$	$1-C_r(0)$
ξ_2^2	$\frac{N^2}{4\langle\vec{J}\rangle^2}\xi_1^2$	$\frac{N^2\xi_1^2}{4\langle J_z\rangle^2}$	$\frac{\xi_1^2}{\langle\sigma_{1z}\rangle^2}$	$\frac{1-C_r(0)}{\langle\sigma_{1z}\rangle_0^2}$
ξ_3^2	$\frac{\lambda_{\min}}{\langle\vec{J}^2\rangle - \frac{N}{2}}$	$\frac{\min\{\xi_1^2,\varsigma^2\}}{4N^{-2}\langle\vec{J}^2\rangle - 2N^{-1}}$	$\frac{\min\{\xi_1^2, 1+\mathcal{C}_{zz}\}}{(1-N^{-1})\langle\vec{\sigma}_1\cdot\vec{\sigma}_2\rangle + N^{-1}}$	$1-C_r(0)$
Concurrence C	$\max(0,\lambda_1-\lambda_2-\lambda_3-\lambda_4)$	$2\max(0,\lvert u\rvert-w, y-\sqrt{v_+v_-})$	$2\max(0,\lvert u\rvert-w, y-\sqrt{v_+v_-})$	C_0

To determine the squeezing parameters and the concurrence, it is convenient to know the correlation function $\mathcal{C}_{zz}$ and the expectation $\langle\vec{\sigma}_1\cdot\vec{\sigma}_2\rangle$, which can be determined from the above expectations as follows:

$$\langle\vec{\sigma}_1\cdot\vec{\sigma}_2\rangle = 1 - spx_0, \tag{53}$$

$$\begin{aligned}\mathcal{C}_{zz} &= s^2\left(\langle\sigma_{1z}\sigma_{2z}\rangle_0 - \langle\sigma_{1z}\rangle_0\langle\sigma_{2z}\rangle_0\right)\\ &= s^2\mathcal{C}_{zz}(0),\end{aligned} \tag{54}$$

where

$$x_0 = 1 + 2\langle\sigma_z\rangle_0 + \langle\sigma_{1z}\sigma_{2z}\rangle_0. \tag{55}$$

Substituting the relevant expectation values and the correlation function into Eqs. (31), (32), and (33) leads to the explicit expression of the spin-squeezing parameters

$$\xi_1^2 = 1 - sC_r(0), \tag{56}$$

$$\xi_2^2 = \frac{\xi_1^2}{(s\langle\sigma_{1z}\rangle_0 - p)^2}, \tag{57}$$

$$\xi_3^2 = \frac{\min\left\{\xi_1^2,\, 1 + s^2\mathcal{C}_{zz}(0)\right\}}{1 + (N^{-1}-1)spx_0}. \tag{58}$$

As the correlation function $\mathcal{C}_{zz}(0) \geqslant 0$, given by Eq. (47), the third parameter can be simplified as

$$\xi_3^2 = \frac{1 - sC_r(0)}{1 + (N^{-1}-1)spx_0}. \tag{59}$$

Initially, the state is spin squeezed, i.e., $\xi_1^2(0) < 1$ or $C_r(0) > 0$. From Eq. (56), one can find that $\xi_1^2 < 1$, except in the asymptotic limit of $p = 1$. As we will see below, for the PDC and DPC,

$$\xi_1^2 = 1 - s^2C_r(0).$$

Thus, we conclude that according to ξ_1^2, the initially spin-squeezed state is always squeezed for $p \neq 1$, irrespective of both the decoherence strength and decoherence models. In other words, there exists no SSSD if we quantify spin squeezing by the first parameter ξ_1^2.

2. *Concurrence*

In the expression (43) of the concurrence, there are three terms inside the max function. The expression can be simplified to (see Appendix E for details):

$$C_r = 2(N-1)\max(0, |u| - w). \tag{60}$$

By using Eqs. (40) and (52c), one finds

$$2(|u| - w) = 2s|u_0| + \frac{s}{2}\left[s - 2 + s\langle\sigma_{1z}\sigma_{2z}\rangle_0 - 2p\langle\sigma_{1z}\rangle_0\right]) \tag{61}$$

$$= sC_0 - \frac{spx_0}{2}. \tag{62}$$

So, we obtain the evolution of the rescaled concurrence as

$$C_r = \max[0, sC_r(0) - 2^{-1}(N-1)spx_0], \tag{63}$$

which depends on the initial concurrence, expectation $\langle\sigma_{1z}\rangle_0$, and correlation $\langle\sigma_{1z}\sigma_{2z}\rangle_0$.

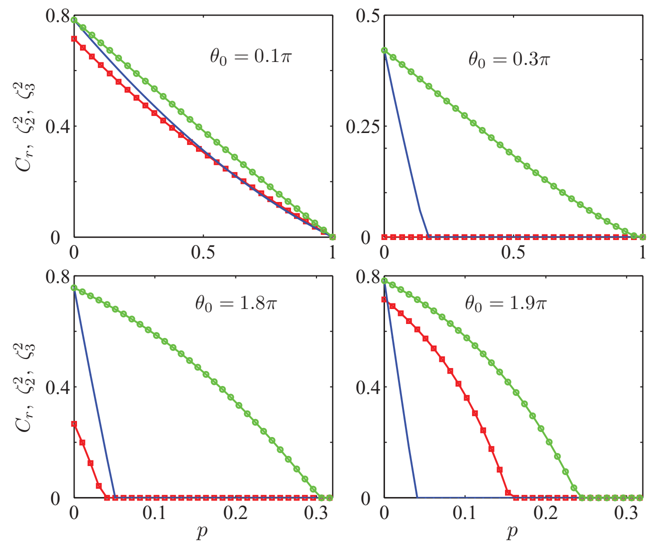

FIG. 1. (Color online) Spin-squeezing parameters ξ_2^2 (red curve with squares), ξ_3^2 (top green curve with circles), and the concurrence C_r (blue solid curve) versus the decoherence strength $p = 1 - \exp(-\gamma t)$ for the amplitude-damping channel, where γ is the damping rate. Here, θ_0 is the initial twist angle given by Eq. (4). In all figures, we consider an ensemble of $N = 12$ spins. Note that for a small initial twist angle θ_0 (e.g., $\theta_0 = 0.1\pi$), the two squeezing parameters and the concurrence all concur. For larger values of θ_0, the parameters ξ_2^2, ξ_3^2, and C become quite different and all vanish for sufficiently large values of the decoherence strength.

3. *Numerical results*

The numerical results for the squeezing parameters and concurrence are shown in Fig. 1 for different initial values of the twist angle θ_0, defined in Eq. (4). For the smaller value of θ_0, e.g., $\theta_0 = \pi/10$, we see that there is no ESD and SSSD. All the spin squeezing and the pairwise entanglement are completely robust against decoherence. Intuitively, the larger is the squeezing, the larger is the vanishing time. However, here, in contrast to this, no matter how small are the squeezing parameters and concurrence, they vanish only in the asymptotic limit. This results from the complex correlations in the initial state and the special characteristics of the ADC.

For larger values of θ_0, as the decoherence strength p increases, the spin squeezing decreases until it suddenly vanishes, so the phenomenon of SSSD occurs. There exists a critical value p_c, after which there is no spin squeezing. The vanishing time of ξ_3^2 is always larger than those of ξ_2^2 and the concurrence. We note that depending on the initial state, the concurrence can vanish before or after ξ_2^2. This means that in our model, the parameter $\xi_3^2 < 1$ implies the existence of pairwise entanglement, while ξ_2^2 does not.

4. *Decoherence strength p_c corresponding to the SSSD*

From Eqs. (57), (58), and (63), the critical value p_c can be analytically obtained as

$$p_c^{(k)} = \frac{x_kC_r(0)}{(N-1)x_0}, \quad (k = 1, 3) \tag{64}$$

$$p_c^{(2)} = \frac{\langle\sigma_{1z}\rangle_0^2 + C_r(0) - 1}{1 + 2\langle\sigma_{1z}\rangle_0 + \langle\sigma_z\rangle_0^2}, \tag{65}$$

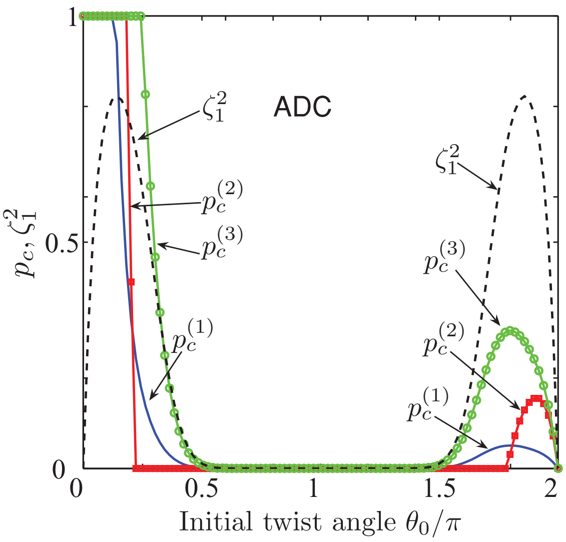

FIG. 2. (Color online) Critical values of the decoherence strength $p_c^{(1)}$ (blue solid curve), $p_c^{(2)}$ (red curve with squares), $p_c^{(3)}$ (green curve with circles), and the squeezing parameter ζ_1^2 (black dashed curve) versus the initial twist angle θ_0 given by Eq. (4) for the amplitude-damping channel, ADC. Here, p_c is related to the vanishing time t_v via $p_c = 1 - \exp(-\gamma t_v)$. At vanishing times, SSSD occurs. The critical values $p_c^{(1)}$, $p_c^{(2)}$, and $p_c^{(3)}$ correspond to the concurrence, squeezing parameter ζ_2^2, and ζ_3^2, respectively.

where $x_1 = 2$ for the concurrence and $x_3 = N$ for the squeezing parameter ζ_3^2. The critical value $p_c^{(2)}$ is for the second squeezing parameter ζ_2^2. Here, p_c is related to the vanishing time t_v via $p_c = 1 - \exp(-\gamma t_v)$.

In Fig. 2, we plot the critical values p_c of the decoherence strength versus θ_0. The initial-state squeezing parameter ζ_1^2 is also plotted for comparison. For a range of small values of θ_0, the entanglement and squeezing are robust to decoherence. The concurrence and parameter ζ_2^2 intersect. However, we do not see the intersections between ζ_3^2 and ζ_2^2 or between ζ_3^2 and the concurrence. We also see that for the same degree of squeezing, the vanishing times are quite different, which implies that except for the spin-squeezing correlations, other type of correlations exist. For large enough initial twist angles $\pi \leqslant \theta_0 \leqslant 2\pi$, the behavior of the squeezing parameter ξ_1^2 is similar to those corresponding to $p_c^{(1)}$ and $p_c^{(3)}$.

C. Phase-damping channel

1. Squeezing parameters and concurrence

Now, we study the spin squeezing and pairwise entanglement under the PDC. For this channel, the expectation values $\langle \sigma_z^{\otimes n} \rangle$ are unchanged and the two correlations $\langle \sigma_{1-}\sigma_{2-} \rangle$ and $\langle \sigma_{1+}\sigma_{2-} \rangle$ evolve as (see Appendix D for details)

$$\begin{aligned}\langle \sigma_{1-}\sigma_{2-} \rangle &= s^2 \langle \sigma_{1-}\sigma_{2-} \rangle,\\ \langle \sigma_{1+}\sigma_{2-} \rangle &= s^2 \langle \sigma_{1+}\sigma_{2-} \rangle.\end{aligned} \tag{66}$$

From the above equations and the fact $\langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle_0 = 1$, one finds

$$\begin{aligned}\langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle &= s^2 \langle \sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} \rangle_0 + \langle \sigma_{1z}\sigma_{2z} \rangle_0\\ &= s^2 (1 - \langle \sigma_{1z}\sigma_{2z} \rangle_0) + \langle \sigma_{1z}\sigma_{2z} \rangle_0,\end{aligned} \tag{67}$$

$$\mathcal{C}_{zz}(p) = \mathcal{C}_{zz}(0). \tag{68}$$

Therefore, from the above properties, we obtain the evolution of the squeezing parameters,

$$\xi_1^2 = 1 - s^2 C_r(0), \tag{69}$$

$$\xi_2^2 = \frac{\xi_1^2}{\langle \sigma_{1z} \rangle_0^2}, \tag{70}$$

and the third parameter becomes

$$\xi_3^2 = \frac{N \min\left[\xi_1^2, 1 + \mathcal{C}_{zz}(0)\right]}{(N-1)[s^2 + (1-s^2)\langle \sigma_{1z}\sigma_{2z} \rangle_0] + 1} \tag{71}$$

$$= \frac{N \xi_1^2}{(N-1)[s^2 + (1-s^2)\langle \sigma_{1z}\sigma_{2z} \rangle_0] + 1}. \tag{72}$$

where we have used Eqs. (67) and (68), and the property $\mathcal{C}_{zz}(0) \geqslant 0$.

From Eq. (66) and the simplified form of the concurrence given by Eq. (60), the concurrence is found to be

$$\begin{aligned}C_r &= \max\{0, 2(N-1)[s^2|u_0| - 4^{-1}(1 - \langle \sigma_{1z}\sigma_{2z} \rangle_0)]\}\\ &= \max\left[0, s^2 C_r(0) + \frac{a_0(s^2-1)}{2}\right],\end{aligned} \tag{73}$$

where

$$a_0 = (N-1)(1 - \langle \sigma_{1z}\sigma_{2z} \rangle_0). \tag{74}$$

Thus, we obtained all time evolutions of the spin-squeezing parameters and the concurrence. To study the phenomenon of SSSD, we below examine the vanishing times.

2. Decoherence strength p_c corresponding to the SSSD

The critical decoherence strengths p_c can be obtained from Eqs. (70), (71), and (73) as follows:

$$p_c^{(k)} = 1 - \left[\frac{a_0}{x_k C_r(0) + a_0}\right]^{\frac{1}{2}}, \tag{75}$$

$$p_c^{(2)} = 1 - \left[\frac{1 - \langle \sigma_{1z} \rangle_0^2}{C_r(0)}\right]^{\frac{1}{2}}, \tag{76}$$

where $k = 1, 3$ and $x_1 = 2$, $x_3 = N$.

In Fig. 3, we plot the decoherence strength p_c versus the twist angle θ_0 of the initial state for the PDC. For this decoherence channel, the critical value $p_c's$ first decrease until they reach zero. Also, it is symmetric with respect to $\theta_0 = \pi$, which is in contrast to the ADC. There are also intersections between the concurrence and parameter ξ_2^2, and the critical value $p_c^{(3)}$ is always larger than $p_c^{(1)}$ and $p_c^{(2)}$.

D. Depolarizing channel

1. Squeezing parameters and concurrence

The decoherence of the squeezing parameter defined by Sørensen *et al.* [18] has been studied in Ref. [27] for the DPC. It is intimately related to the second squeezing parameter ξ_2^2. For the DPC, the evolution of correlations $\langle \sigma_{1-}\sigma_{2-} \rangle$ and $\langle \sigma_{1+}\sigma_{2-} \rangle$ are the same as those of the DPC given by Eq. (66), and the expectations $\langle \sigma_{1z} \rangle$ and $\langle \sigma_{1z}\sigma_{2z} \rangle$ change as (see Appendix D).

$$\langle \sigma_{1z} \rangle = s \langle \sigma_{1z} \rangle_0, \tag{77}$$

$$\langle \sigma_{1z}\sigma_{2z} \rangle = s^2 \langle \sigma_{1z}\sigma_{2z} \rangle_0. \tag{78}$$

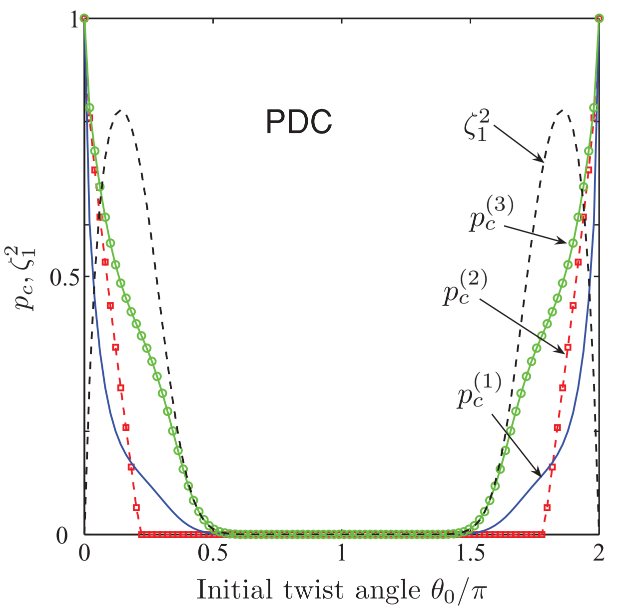

FIG. 3. (Color online) Same as in Fig. 2 but for the phase-damping channel, PDC, instead of ADC.

From these equations, we further have

$$\langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle = s^2 \langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle_0 = s^2, \tag{79}$$

$$\mathcal{C}_{zz} = s^2 \left(\langle \sigma_{1z}\sigma_{2z} \rangle_0 - \langle \sigma_{1z} \rangle_0 \langle \sigma_{2z} \rangle_0 \right) = s^2 \mathcal{C}_{zz}(0). \tag{80}$$

The squeezing parameter ξ_1^2 is given by Eq. (69), and the other two squeezing parameters are obtained as

$$\xi_2^2 = \frac{\xi_1^2}{s^2 \langle \sigma_{1z} \rangle_0^2}, \tag{81}$$

$$\xi_3^2 = \frac{N \min\left\{\xi_1^2, 1 + s^2 \mathcal{C}_{zz}(0)\right\}}{(N-1)s^2 + 1} = \frac{N\xi_1^2}{(N-1)s^2 + 1}. \tag{82}$$

By making use of Eqs. (66) and (78) and starting from the simplified form of the concurrence (60), we obtain

$$\begin{aligned} C_r &= \max\left\{0, 2(N-1)\left[s^2|u_0| - \tfrac{1}{4}(1 - s^2\langle \sigma_{1z}\sigma_{2z} \rangle_0)\right]\right\} \\ &= \max[0, s^2 C_r(0) + 2^{-1}(N-1)(s^2 - 1)]. \end{aligned} \tag{83}$$

We observe that the concurrence is dependent only on the initial value itself, not other ones.

2. *Decoherence strength p_c corresponding to the SSSD*

From Eqs. (83), (81), and (82), the vanishing times are analytically calculated as

$$p_v^{(k)} = 1 - \left[\frac{N-1}{x_k C_r(0) + N - 1}\right]^{\frac{1}{2}}, \tag{84}$$

$$p_v^{(2)} = 1 - \left[\frac{1}{C_r(0) + \langle \sigma_{1z} \rangle_0^2}\right]^{\frac{1}{2}}, \tag{85}$$

where $k = 1, 3$ and $x_1 = 2$, $x_3 = N$.

TABLE II. Analytical results for the time-evolutions of all relevant expectations, correlations, spin-squeezing parameters, and concurrence, as well as the critical values p_c of the decoherence strength p. This is done for the three decoherence channels considered in this work. For the concurrence C, we give the expression for C_r', which is related to the rescaled concurrence C_r via $C_r = \max(0, C_r')$.

	Amplitude-damping channel (ADC)	Phase-damping channel (PDC)	Depolarizing channel (DPC)
$\langle \sigma_{1z} \rangle$	$s\langle \sigma_{1z} \rangle_0 - p$	$\langle \sigma_{1z} \rangle_0$	$s\langle \sigma_{1z} \rangle_0$
$\langle \sigma_{1z}\sigma_{2z} \rangle$	$s^2\langle \sigma_{1z}\sigma_{2z} \rangle_0 - 2sp\langle \sigma_{1z} \rangle_0 + p^2$	$\langle \sigma_{1z}\sigma_{2z} \rangle_0$	$s^2\langle \sigma_{1z}\sigma_{2z} \rangle_0$
$\langle \sigma_{1+}\sigma_{2-} \rangle$	$s\langle \sigma_{1+}\sigma_{2-} \rangle_0$	$s^2\langle \sigma_{1+}\sigma_{2-} \rangle_0$	$s^2\langle \sigma_{1+}\sigma_{2-} \rangle_0$
$\langle \sigma_{1-}\sigma_{2-} \rangle$	$s\langle \sigma_{1-}\sigma_{2-} \rangle_0$	$s^2\langle \sigma_{1-}\sigma_{2-} \rangle_0$	$s^2\langle \sigma_{1-}\sigma_{2-} \rangle_0$
$\langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle$	$1 - spx_0$	$s^2(1 - \langle \sigma_{1z}\sigma_{2z} \rangle_0) + \langle \sigma_{1z}\sigma_{2z} \rangle_0$	s^2
$\mathcal{C}_{zz}$	$s^2\mathcal{C}_{zz}(0)$	$\mathcal{C}_{zz}(0)$	$s^2\mathcal{C}_{zz}(0)$
ξ_1^2	$1 - sC_r(0)$	$1 - s^2C_r(0)$	$1 - s^2C_r(0)$
ξ_2^2	$\frac{1 - sC_r(0)}{(s\langle \sigma_{1z} \rangle_0 - p)^2}$	$\frac{1 - s^2C_r(0)}{\langle \sigma_{1z} \rangle_0^2}$	$\frac{1 - s^2C_r(0)}{s^2\langle \sigma_{1z} \rangle_0^2}$
ξ_3^2	$\frac{1 - sC_r(0)}{1 + (N^{-1} - 1)spx_0}$	$\frac{1 - s^2C_r(0)}{(1 - N^{-1})[s^2 + (1 - s^2)\langle \sigma_{1z}\sigma_{2z} \rangle_0] + N^{-1}}$	$\frac{1 - s^2C_r(0)}{(1 - N^{-1})s^2 + N^{-1}}$
C_r'	$sC_r(0) - (N-1)spx_0/2$	$s^2C_r(0) + a_0(s^2 - 1)/2$	$s^2C_r(0) + (N-1)(s^2 - 1)/2$
$p_c^{(1)}$	$\frac{2C_r(0)}{(N-1)x_0}$	$1 - \left(\frac{a_0}{2C_r(0) + a_0}\right)^{\frac{1}{2}}$	$1 - \left(\frac{N-1}{2C_r(0) + N - 1}\right)^{\frac{1}{2}}$
$p_c^{(2)}$	$\frac{\langle \sigma_{1z} \rangle_0^2 + C_r(0) - 1}{1 + 2\langle \sigma_{1z} \rangle_0 + \langle \sigma_z \rangle_0^2}$	$1 - \left(\frac{1 - \langle \sigma_{1z} \rangle_0^2}{C_r(0)}\right)^{\frac{1}{2}}$	$1 - \left(\frac{1}{C_r(0) + \langle \sigma_{1z} \rangle_0^2}\right)^{\frac{1}{2}}$
$p_c^{(3)}$	$\frac{NC_r(0)}{(N-1)x_0}$	$1 - \left(\frac{a_0}{NC_r(0) + a_0}\right)^{\frac{1}{2}}$	$1 - \left(\frac{N-1}{NC_r(0) + N - 1}\right)^{\frac{1}{2}}$

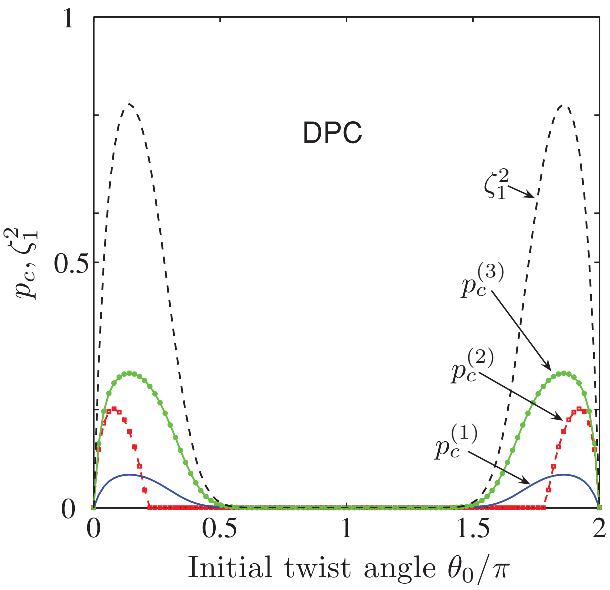

FIG. 4. (Color online) Same as in Fig. 2 but for the depolarizing channel, DPC, instead of ADC.

In Fig. 3, we plot the critical values p_c versus the initial twist angle θ_0 for the DPC. For the DPC, the $p_c^{\prime}s$ first increase until they reach their maxima and then decrease to zero. Also, it is symmetric with respect to $\theta_0 = \pi$, which is the same as for the PDC. There are also intersections between the concurrence and the parameter ξ_2^2. Qualitatively, the behaviors of $p_c^{(1)}$ and $p_c^{(3)}$ are the same as that of the squeezing parameter ζ_1^2. This implies that the larger the squeezing, the larger is the critical value p_c.

The common features of these three decoherence channels are: (i) The critical value p_{v3} is always larger or equal than the other two, namely the spin-squeezing correlations according to ξ_3^2 are more robust; (ii) there always exist two intersections between the concurrence and the parameter ξ_2^2, for θ_0 from 0 to 2π, irrespective of the decoherence channels; (iii) when there is no squeezing (central area of Figs. 2, 3, and 4), all vanishing times are zero. Table II conveniently lists all the analytical results obtained in this section.

VI. CONCLUSIONS AND REMARKS

TO summarize, for a spin ensemble in a typical spin-squeezing initial state under three different decoherence channels, we have studied spin squeezing with three different parameters in comparison with the pairwise entanglement quantified by the concurrence. When the subsystems of the correlated system decay asymptotically in time, the spin-squeezing parameter ζ_1^2 also decays asymptotically in time for all three types of decoherence. However, for the other two squeezing parameters ζ_2^2 and ζ_3^2, we find the appearance of spin-squeezing sudden death and entanglement sudden death. The global behaviors of the correlated state are markedly different from the local ones. The spin-squeezing parameter ζ_2^2 can vanish before, simultaneously, or after the concurrence, while the squeezing parameter ζ_3^2 is always the last to vanish. This means that this parameter is more robust to decoherence, and it can detect more entanglement than ξ_2^2.

Our analytical approach for the vanishing times can be applied to any initial quantum correlated states, not restricted to the present one-axis twisted state. Moreover, for more complicated channels, such as the amplitude-damping channel at finite temperatures [31] or the channel discussed in Ref. [65], the method developed in this article can be readily applied to study spin squeezing under these decoherence channels.

Our investigations show the widespread occurrence of sudden death phenomena in many-body quantum correlations. Since there exists different vanishing times for different squeezing parameters, spin squeezing offers a possible way to detect the total spin correlation and their quantum fluctuations with distinguishable time scales. The discovery of different lifetimes for various spin-squeezing parameters means that, in some time region, there still exists another quantum correlation when other quantum correlations suddenly vanish. However, to determine which kind of correlations will vanish, one possible approach is to further invoke irreducible multiparty correlations [66], where the multipartite correlations are classified in a series of irreducible k-party ones. If we could obtain the time evolution behaviors of such irreducible multipartite correlations in various decoherence channels, we could classify lifetimes for the spin-squeezing sudden death of various multipartite correlations order by order.

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APPENDIX A: SPIN-SQUEEZING PARAMETER ξ_3^2 FOR STATES WITH PARITY SYMMETRY

Here, we calculate the spin-squeezing parameter ξ_3^2 for collective states with either even or odd parity symmetry. For such states, we immediately have

$$\langle J_x\rangle = \langle J_y\rangle = \langle J_x J_z\rangle = \langle J_y J_z\rangle = 0 \tag{A1}$$

as the operators change the parity of the state. Then, the mean spin direction is along the z direction and the correlation matrix given by Eq. (24) is simplified to

$$\mathbf{C} = \begin{pmatrix} \langle J_x^2\rangle & C_{xy} & 0 \\ C_{xy} & \langle J_y^2\rangle & 0 \\ 0 & 0 & \langle J_z^2\rangle \end{pmatrix}, \tag{A2}$$

where $C_{xy} = \langle [J_x, J_y]_+\rangle/2$. From the correlation matrix $\mathbf{C}$ and the definition of covariance matrix γ given by Eq. (23), one finds

$$\Gamma = \begin{pmatrix} N\langle J_x^2\rangle & NC_{xy} & 0 \\ NC_{xy} & N\langle J_y^2\rangle & 0 \\ 0 & 0 & N(\Delta J_z)^2 + \langle J_z^2\rangle \end{pmatrix}. \tag{A3}$$

This matrix has a block-diagonal form and the eigenvalues of the 2×2 block are obtained as

$$\lambda_\pm = \frac{N}{2}\left(\langle J_x^2 + J_y^2\rangle \pm |\langle J_-^2\rangle|\right). \tag{A4}$$

In deriving the above equation, we have used the relation

$$J_-^2 = J_x^2 - J_y^2 - i[J_x, J_y]_+. \tag{A5}$$

Therefore, the smallest eigenvalue $\lambda_{\min}$ of Γ is obtained as

$$\lambda_{\min} = \min\left(\lambda_-, N(\Delta J_z)^2 + \langle J_z^2\rangle\right), \tag{A6}$$

where λ_- differs from the squeezing parameter ξ_1^2 given by Eq. (25) by only a multiplicative constant, as seen by comparing Eqs. (25) and (A6). From Eqs. (A6) and (21), one finds that the squeezing parameter ξ_3^2 is given by Eq. (27).

APPENDIX B: SPIN-SQUEEZING PARAMETERS FOR THE ONE-AXIS TWISTED STATE

Here, we will use the Heisenberg picture to derive the relevant expectations and spin-squeezing parameters for the initial state [67,68]. To determine the spin-squeezing parameter ξ_1^2 given by Eq. (31), one needs to know the expectation $\langle\sigma_{1z}\rangle_0$, and correlations $\langle\sigma_{1+}\sigma_{2-}\rangle_0$ and $\langle\sigma_{1-}\sigma_{2-}\rangle_0$. We first consider the expectation $\langle\sigma_{1z}\rangle_0$. For simplicity, we omit the subscript 0 in the following formulas.

1. Expectation $\langle\sigma_{1z}\rangle$

The evolution operator can be written as,

$$U = \exp\left(-i\chi t J_x^2\right) = \exp\left(-i\theta \sum_{k>l} j_{kx} j_{lx}\right) \tag{B1}$$

up to a trivial phase, where $\theta = 2\chi t$ given by Eq. (4). From this form, the evolution of j_{1z} can be obtained as

$$U^\dagger j_{1z} U = j_{1z}\cos\left[\theta j_x^{(2)}\right] + j_{1y}\sin\left[\theta j_x^{(2)}\right], \tag{B2}$$

where

$$j_x^{(k)} = \sum_{l=k}^{N} j_{lx}. \tag{B3}$$

Therefore, the expectations are

$$\langle j_{1z}\rangle = -2^{-1}\langle \mathbf{1}'|\cos\left[\theta j_x^{(2)}\right]|\mathbf{1}'\rangle \tag{B4}$$

since $\langle 1|j_{1y}|1\rangle = 0$. Here, $|\mathbf{1}'\rangle = |1\rangle_2 \otimes \cdots \otimes |1\rangle_N$. So, one can find the following form for the expectation values

$$\begin{aligned}\langle\mathbf{1}|\cos[\theta J_x]|\mathbf{1}\rangle &= (\langle\mathbf{1}|e^{i\theta J_x}|\mathbf{1}\rangle + \text{c.c.})/2 \\ &= \left(\Pi_{k=1}^N\langle 1|e^{i\theta j_{kx}}|1\rangle + \text{c.c.}\right)/2 \\ &= \cos^N(\theta'),\end{aligned} \tag{B5}$$

where $\theta' = \theta/2$ and $|\mathbf{1}\rangle = |1\rangle^{\otimes N}$.

By using Eqs. (B4) and (B5), one gets

$$\langle\sigma_z\rangle = -\cos^{N-1}(\theta'). \tag{B6}$$

2. Correlation $\langle\sigma_{1+}\sigma_{2-}\rangle$

Since the operator $\sigma_{1x}\sigma_{2x}$ commutes with the unitary operator U, we easily obtain

$$\langle\sigma_{1x}\sigma_{2x}\rangle = 0. \tag{B7}$$

We now compute the correlations $\langle\sigma_{1z}\sigma_{2z}\rangle$. From the unitary operator,

$$\begin{aligned}&U^\dagger j_{1z} j_{2z} U \\ &= \left[j_{1z}\cos\left(\theta j_x^{(2)}\right) + j_{1y}\sin\left(\theta j_x^{(2)}\right)\right] \\ &\quad\times\left[j_{2z}\cos\left[\theta\left(j_{1x} + j_x^{(3)}\right)\right] + j_{2y}\sin\left[\theta\left(j_{1x} + j_x^{(3)}\right)\right]\right] \\ &= \left[j_{1z}\cos(\theta j_{2x})\cos\left(\theta j_x^{(3)}\right) - j_{1z}\sin(\theta j_{2x})\sin\left(\theta j_x^{(3)}\right)\right. \\ &\quad\left. + j_{1y}\sin(\theta j_{2x})\cos\left(\theta j_x^{(3)}\right) + j_{1y}\cos(\theta j_{2x})\sin\left(\theta j_x^{(3)}\right)\right] \\ &\quad\times\left[j_{2z}\cos(\theta j_{1x})\cos\left(\theta j_x^{(3)}\right) - j_{2z}\sin(\theta j_{1x})\sin\left(\theta j_x^{(3)}\right)\right. \\ &\quad\left. + j_{2y}\sin(\theta j_{1x})\cos\left(\theta j_x^{(3)}\right) + j_{2y}\cos(\theta j_{1x})\sin\left(\theta j_x^{(3)}\right)\right].\end{aligned}$$

Although there are 16 terms after expanding the above equation, only 4 terms survive when calculating $\langle s_{1z}s_{2z}\rangle$. We then have

$$\begin{aligned}\langle j_{1z} j_{2z}\rangle &= \langle\mathbf{1}|\, j_{1z} j_{2z}\cos^2(\theta/2)\cos^2\left(\theta j_x^{(3)}\right) \\ &\quad - j_{1z} j_{2x} j_{2y}\sin(\theta)\sin^2\left(\theta j_x^{(3)}\right) \\ &\quad + 4 j_{1y} j_{1x} j_{2x} j_{2y}\sin^2(\theta/2)\cos^2\left(\theta j_x^{(3)}\right) \\ &\quad - j_{1y} j_{1x} j_{2z}\sin(\theta)\sin^2\left(\theta j_x^{(3)}\right)|\mathbf{1}\rangle \\ &= 4^{-1}\langle\mathbf{1}'|\cos^2\left(\theta j_x^{(3)}\right)|\mathbf{1}'\rangle \\ &= 8^{-1}\langle\mathbf{1}'|\left[1 + \cos\left(2\theta j_x^{(3)}\right)\right]|\mathbf{1}'\rangle \\ &= 8^{-1}[1 + \cos^{N-2}(\theta)],\end{aligned} \tag{B8}$$

where $|\mathbf{1}'\rangle = |1\rangle_3 \otimes \ldots \otimes |1\rangle_N$. The second equality in Eq. (B8) is due to the property $j_x j_y = -j_y j_x = i j_z/2$, and the last equality from Eq. (B5). Finally, from the above equation, one finds

$$\langle\sigma_{1z}\sigma_{2z}\rangle = 2^{-1}(1 + \cos^{N-2}\theta). \tag{B9}$$

Due to the relation $\langle\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z}\rangle = 1$ for the initial state, the correlation $\langle\sigma_{1y}\sigma_{2y}\rangle$ is obtained from Eqs. (B7) and (B9) as

$$\langle\sigma_{1y}\sigma_{2y}\rangle = 2^{-1}(1 - \cos^{N-2}\theta). \tag{B10}$$

Substituting Eqs. (B7) and (B10) into the following relations

$$\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} = 2(\sigma_{1+}\sigma_{2-} + \sigma_{1-}\sigma_{2+})$$

leads to one element of the two-spin reduced density matrix,

$$y_0 = \langle\sigma_{1+}\sigma_{2-}\rangle = 8^{-1}(1 - \cos^{N-2}\theta), \tag{B11}$$

where the relation $\langle\sigma_{1+}\sigma_{2-}\rangle = \langle\sigma_{1-}\sigma_{2+}\rangle$ is used due to the exchange symmetry.

3. Correlation $\langle\sigma_{1-}\sigma_{2-}\rangle$

To calculate the correlation $\langle\sigma_{1-}\sigma_{2-}\rangle$, due to the following relations

$$\sigma_{1x}\sigma_{2x} - \sigma_{1y}\sigma_{2y} = 2(\sigma_{1+}\sigma_{2+} + \sigma_{1-}\sigma_{2-}), \tag{B12}$$
$$i(\sigma_{1x}\sigma_{2y} + \sigma_{1y}\sigma_{2x}) = 2(\sigma_{1+}\sigma_{2+} - \sigma_{1-}\sigma_{2-}), \tag{B13}$$

we need to know the expectations $\langle j_{1x}j_{2y}\rangle$. The evolution of $j_{1x}j_{2y}$ is given by

$$\begin{aligned}U^\dagger s_{1x}s_{2y}U = {}& j_{1x}\left\{j_{2y}\cos\left[\theta\left(j_{1x}+j_x^{(3)}\right)\right]\right.\\ &\left.- j_{2z}\sin\left[\theta\left(j_{1x}+j_x^{(3)}\right)\right]\right\},\end{aligned}$$

and the expectation is obtained as

$$\begin{aligned}\langle j_{1x}j_{2y}\rangle &= 2^{-1}\langle\mathbf{1}'|j_{1x}\sin\left[\theta\left(j_{1x}+j_x^{(3)}\right)\right]|\mathbf{1}'\rangle\\ &= (4i)^{-1}\langle\mathbf{1}'|j_{1x}e^{i\theta j_{1x}}\Pi_{k=3}^N e^{i\theta j_{kx}}\\ &\quad - j_{1x}e^{-i\theta j_{1x}}\Pi_{k=3}^N e^{-i\theta j_{kx}}|\mathbf{1}'\rangle\\ &= (4i)^{-1}\cos^{N-2}(\theta')\langle 1|j_{1x}e^{i\theta j_{1x}} - j_{1x}e^{-i\theta j_{1x}}|1\rangle\\ &= 2^{-1}\cos^{N-2}(\theta')\langle 1|j_{1x}\sin(\theta j_{1x})|1\rangle\\ &= 4^{-1}\sin(\theta')\cos^{N-2}(\theta').\end{aligned}$$

Here, $|\mathbf{1}'\rangle = |1\rangle_1 \otimes |1\rangle_3 \otimes \cdots \otimes |1\rangle_N$, where $|1\rangle_2$ is absent. Moreover, $\langle j_{1y}j_{2x}\rangle = \langle j_{1x}j_{2y}\rangle$ due to the exchange symmetry, and thus,

$$\langle j_{1x}j_{2y} + j_{1y}j_{2x}\rangle = 2^{-1}\sin(\theta')\cos^{N-2}(\theta').$$

For the initial state (2), we obtain the following expectations [16,63]

$$\langle\sigma_{1x}\sigma_{2y} + \sigma_{1y}\sigma_{2x}\rangle = 2\sin(\theta')\cos^{N-2}(\theta'). \tag{B14}$$

The combination of Eqs. (B7), (B10), (B12), (B13), and (B14) leads to the correlation

$$\begin{aligned}u_0 = \langle\sigma_{1-}\sigma_{2-}\rangle &= -8^{-1}(1-\cos^{N-2}\theta)\\ &\quad -i2^{-1}\sin(\theta')\cos^{N-2}(\theta').\end{aligned} \tag{B15}$$

Substituting Eqs. (B11) and (B15) to Eq. (31) leads to the expression of the squeezing parameter ξ_1^2 given by Eq. (44).

APPENDIX C: PROOF OF $\mathcal{C}_{zz}(0) \geqslant 0$

To prove this, we will not use this specific function of the initial twist angle θ as given by Eq. (47), but use the positivity of the reduced density matrix (38). We first notice an identity

$$\mathcal{C}_{zz} = 4(v_+v_- - w^2),$$

which results from Eqs. (39) and (40). This is a key step. Also there exists another identity

$$w_0 = y_0 \tag{C1}$$

as $\langle\vec{\sigma}_1\cdot\vec{\sigma}_2\rangle_0 = 1$. From the positivity of the reduced density matrix (38), one has

$$v_{0+}v_{0-} \geqslant |u_0|^2 \geqslant y_0^2 = w_0^2,$$

where the second inequality follows from Eq. (40) and the last equality results from Eq. (C1). This completes the proof.

APPENDIX D: DERIVATION OF THE EVOLUTION OF THE CORRELATIONS AND EXPECTATIONS UNDER DECOHERENCE

For an arbitrary matrix

$$A = \begin{pmatrix} a & b\\ c & d\end{pmatrix},$$

from the Kraus operators (12) for the ADC, it is straightforward to find

$$\mathcal{E}(A) = \begin{pmatrix} sa & \sqrt{s}b\\ \sqrt{s}c & d+pa\end{pmatrix},$$
$$\mathcal{E}^\dagger(A) = \begin{pmatrix} sa+pd & \sqrt{s}b\\ \sqrt{s}c & d\end{pmatrix}.$$

The above equations imply that

$$\mathcal{E}^\dagger(\sigma_\mu) = \sqrt{s}\sigma_\mu \quad \text{for} \quad \mu = x, y,$$
$$\mathcal{E}^\dagger(\sigma_z) = s\sigma_z - p.$$

As we considered independent and identical decoherence channels acting separately on each spin, the evolution correlations and expectations in Eqs. (52b), (52c), and (52d) are obtained directly from the above equations.

From the Kraus operators (14), the evolution of the matrix A under the PDC is obtained as

$$\mathcal{E}(A) = \mathcal{E}^\dagger(A) = \begin{pmatrix} a & sb\\ sc & d\end{pmatrix},$$

from which one finds

$$\mathcal{E}^\dagger(\sigma_\mu) = s\sigma_\mu \quad \text{for} \quad \mu = x, y$$

$$\mathcal{E}^\dagger(\sigma_z) = \sigma_z.$$

So expectations $\langle\sigma_z^{\otimes n}\rangle$ are unchanged and Eq. (66) is obtained.

From the Kraus operators (16) of the DPC, the evolution of the matrix A is given by

$$\begin{aligned}\mathcal{E}(A) &= \mathcal{E}^\dagger(A)\\ &= \begin{pmatrix} as + \frac{p}{2}(a+d) & sb\\ sc & ds + \frac{p}{2}(a+d)\end{pmatrix}\end{aligned}$$

from which one finds

$$\mathcal{E}^\dagger(\sigma_\alpha) = s\sigma_\alpha \quad \text{for} \quad \alpha \in \{x, y, z\}.$$

Then, Eq. (78) is obtained.

APPENDIX E: SIMPLIFIED FORM OF THE CONCURRENCE

For our three kinds of decoherence channels, the concurrence (43) can be simplified and given by

$$\begin{aligned}C &= \max\{0, 2(|u|-w), 2(y-\sqrt{v_+v_-})\}\\ &= \max\{0, 2(|u|-w)\}.\end{aligned} \tag{E1}$$

If one can prove

$$|u| - y \geqslant 0, \tag{E2}$$

$$w - \sqrt{v_+v_-} \leqslant 0, \tag{E3}$$

then we obtain the simplified form shown in Eq. (E1). The last inequality can be replaced by

$$w^2 - v_+ v_- \leqslant 0 \tag{E4}$$

as w and $v_+ v_-$ are real.

We first consider the ADC channel. From Eqs. (52b), (52c), and (54), one obtains

$$|u| - y = s(|u_0| - y_0) \geqslant 0, \tag{E5}$$

$$w^2 - v_+ v_- = -\frac{1}{4}\mathcal{C}_{zz} = -\frac{s^2}{4}\mathcal{C}_{zz}(0) \leqslant 0. \tag{E6}$$

where the inequalities result from Eqs. (44) and (47), respectively. So, the inequality (E4) follows.

For the PDC, from Eq. (66) and fact that $\langle \sigma_z^{\otimes n} \rangle$ is unchanged under decoherence, the concurrence can also be simplified due to the following properties:

$$|u| - y = s^2(|u_0| - y_0) \geqslant 0,$$
$$w^2 - v_+ v_- = -\tfrac{1}{4}\mathcal{C}_{zz}(0) \leqslant 0.$$

For the DPC, from Eqs. (66) and (78), one has

$$|u| - y = s^2(|u_0| - y_0) \geqslant 0, \tag{E7}$$

$$w^2 - v_+ v_- = -\frac{s^2}{4}\mathcal{C}_{zz}(0) \leqslant 0. \tag{E8}$$

So, again, the concurrence can be simplified to the form shown in Eq. (E1). This completes the proof.

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