Experimental linear-optical implementation of a multifunctional optimal qubit cloner

SUPPLEMENTARY INFORMATION

Karel Lemr,[1,*] Karol Bartkiewicz,[2,†] Antonín Černoch,[1] Jan Soubusta,[3] and Adam Miranowicz[2]

[1]RCPTM, Joint Laboratory of Optics of Palacký University and Institute of Physics of Academy of Sciences of the Czech Republic, Faculty of Science, Palacký University 17. listopadu 12, 771 46 Olomouc, Czech Republic

[2]Faculty of Physics, Adam Mickiewicz University, PL-61-614 Poznań, Poland

[3]Institute of Physics of Academy of Science of the Czech Republic, Joint Laboratory of Optics of PU and IP AS CR, 17. listopadu 12, 772 07 Olomouc, Czech Republic

(Dated: March 20, 2012)

OPTIMALITY PROOF OF THE CLONERS

Average fidelity of cloning a set of qubits

While considering optimal symmetric $1 \to 2$ cloning the optimized figure of merit is the average single-copy fidelity. We express the average fidelity as $F = \mathrm{Tr}\,(R\chi)$, where the operator χ is isomorphic to a completely-positive trace-preserving map [1] (CPTP) which performs the cloning operation and $R = \langle \rho^{\mathrm{T}} \otimes (\rho \otimes 1 + 1 \otimes \rho)\rangle/2$, where $\rho = |\psi\rangle\langle\psi|$ is the density matrix of a qubit from the cloned set. The matrix R is given explicitly for any phase-covariant cloner as:

$$R = \frac{1}{8}\begin{pmatrix} 8c_2^4 & 0 & 0 & 0 & 0 & s_1^2 & s_1^2 & 0 \\ 0 & 4c_2^2 & 0 & 0 & 0 & 0 & 0 & s_1^2 \\ 0 & 0 & 4c_2^2 & 0 & 0 & 0 & 0 & s_1^2 \\ 0 & 0 & 0 & 2s_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2s_1^2 & 0 & 0 & 0 \\ s_1^2 & 0 & 0 & 0 & 0 & 4s_2^2 & 0 & 0 \\ s_1^2 & 0 & 0 & 0 & 0 & 0 & 4s_2^2 & 0 \\ 0 & s_1^2 & s_1^2 & 0 & 0 & 0 & 0 & 8s_2^4 \end{pmatrix}, \qquad (1)$$

where $s_i^j = \langle \sin^j(\theta/i)\rangle$ and $c_i^j = \langle \cos^j(\theta/i)\rangle$ for $i, j = 1, 2, 3...$, and the angle bracket stands for averaging over the input qubit distribution $g(\theta, \phi)$. We find the optimal cloning map χ by maximising the functional F.

The necessary conditions for optimality of the MPCC

First, let us note that the optimal cloning map χ must satisfy the following symmetry conditions imposed by the symmetry of a mirror phase-covariant set of qubits, i.e., $\forall_{\phi\in[0,2\pi]}[R_z(\phi)^* \otimes R_z(\phi)^{\otimes 2}, \chi] = 0$ and $[\sigma_x^{\otimes 3}, \chi] = 0$. Second, we assume symmetric cloning, and therefore we require that both clones have the same fidelity. Thus, we demand $[\mathbb{1}_{\mathrm{in}} \otimes \mathrm{SWAP}, \chi] = 0$. Moreover, we can show that elements of χ must be real since maximized fidelity depends linearly only on the real part of the off-diagonal elements. We must also remember that χ must preserve trace, i.e., $\mathrm{Tr}_{\mathrm{out}}\,[\chi] = \mathbb{1}$. All the above conditions imply the following form of the map χ being a mixture of two CPTP maps:

$$\chi = (1-p)\begin{pmatrix} \Lambda^2 & 0 & 0 & 0 & 0 & \frac{\Lambda\bar{\Lambda}}{\sqrt{2}} & \frac{\Lambda\bar{\Lambda}}{\sqrt{2}} & 0 \\ 0 & \frac{\bar{\Lambda}^2}{2} & \frac{\bar{\Lambda}^2}{2} & 0 & 0 & 0 & 0 & \frac{\Lambda\bar{\Lambda}}{\sqrt{2}} \\ 0 & \frac{\bar{\Lambda}^2}{2} & \frac{\bar{\Lambda}^2}{2} & 0 & 0 & 0 & 0 & \frac{\Lambda\bar{\Lambda}}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\Lambda\bar{\Lambda}}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{\Lambda^2}{2} & \frac{\Lambda^2}{2} & 0 \\ \frac{\Lambda\bar{\Lambda}}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{\Lambda^2}{2} & \frac{\Lambda^2}{2} & 0 \\ 0 & \frac{\Lambda\bar{\Lambda}}{\sqrt{2}} & \frac{\Lambda\bar{\Lambda}}{\sqrt{2}} & 0 & 0 & 0 & 0 & \Lambda^2 \end{pmatrix} + p(|011\rangle\langle 011| + |100\rangle\langle 100|). \qquad (2)$$

where $\bar{\Lambda}^2 + \Lambda^2 = 1$ and $1 \geq \Lambda,\ p \geq 0$ are free parameters. Since we can write the cloning fidelity as $F = (1-p)F_\Lambda + ps_1^2/2$ it is apparent that in order to achieve maximal fidelity assuming $F > 1/2$ we must set $p = 0$. Now, we can derive the expression for Λ demanding that $\mathrm{d}F/\mathrm{d}\Lambda = 0$. As one of the solutions we obtain

$$\Lambda(c_1^2) = \sqrt{\frac{1}{2} + \frac{c_1^2}{2\sqrt{P}}}, \qquad (3)$$

where $P = 2 - 4c_1^2 + 3c_1^4$. However, at this point we cannot conclude that it is optimal.

The sufficient conditions for optimality of the MPCC

As noted by Audenaert and De Moor [2] the problem of designing an optimal cloning machine can be solved by means of semidefinite programming. Moreover, it was noted that as long as χ is a CPTP map in a convex set, there are only global extrema. It can be shown that χ maximizes fidelity $F = \mathrm{Tr}[R\chi]$ if the following conditions are satisfied:

$$(A - R)\chi = 0, \qquad (4)$$
$$A - R \geq 0, \qquad (5)$$

where $A = \lambda \otimes \mathbb{1} \geq 0$ is a positive semidefinite matrix of Lagrange multipliers ensuring that χ is CPTP map, i.e.,

$\mathrm{Tr}_{\mathrm{out}}(\chi) = \mathbb{1}_{\mathrm{in}}$. The operator $\lambda = \mathrm{Tr}_{\mathrm{out}}(R\chi)$ is derived by demanding that the variance of fidelity F over χ should be equal to zero. If the condition (4) is satisfied, then for any CPTP map χ we obtain $\mathrm{Tr}[(A-R)\chi] \geq 0$. It also follows from the trace preservation condition that $\mathrm{Tr}(A\chi) = \mathrm{Tr}\lambda$. Hence, the fidelity F is bounded by the trace preservation condition and $F \leq \mathrm{Tr}[\lambda]$. If inequality is saturated by χ, then χ represents the optimal cloning transformation.

For MPCC we have

$$\lambda = \frac{1}{4}[(1+c_1^2)\Lambda^2 + \bar{\Lambda}^2 + \sqrt{2}(1-c_1^2)\Lambda\bar{\Lambda}]\,\mathbb{1}_{\mathrm{in}}. \qquad (6)$$

Henceforth, it can be easily shown that $\mathrm{Tr}\lambda - F = 0$. The eigenvalues of operator $\Delta = A - R$ can be expressed in terms of R matrix elements in the following way:

$$\begin{aligned} \delta_1 &= \frac{1}{2}\left(F - \frac{1}{2}\right), \\ \delta_2 &= \frac{1}{2}\left(F - \frac{1-c_1^2}{2}\right), \\ \delta_{3,4} &= \frac{1}{2}\left(F - R_{1,1} - R_{2,2} \pm \bar{R}\right), \end{aligned} \qquad (7)$$

where $\bar{R}^2 = (R_{1,1} - R_{2,2})^2 + 8R_{1,6}^2$. All the eigenvalues are double degenerated. Moreover, we have

$$F = R_{1,1} + R_{2,2} + \bar{R}. \qquad (8)$$

Thus, $\delta_3 = F - (2+c_1^2)/4$ and $\delta_4 = 0$. Since $F > 3/4, \forall_i \delta_i \geq 0$, we conclude that Δ is a positive semidefinite matrix. Thus, we have shown that the conditions (4) and (5) are satisfied, which completes the proof.

COMPENSATING FOR IMPERFECT TRANSMITTANCES

In our case the equation relating beam-splitter transmittances ($\mu + \nu = 1$) does not hold and we have $\mu + \nu \neq 1$. Hence additional filtering operations are required in order to maintain the maximum achievable fidelity of the setup. This additional filtering manifests itself in two ways. First, one needs to unbalance the ancilla-dependent filtering performed by filters F in both BDAs. We require $\tau_1 = \tau$ and $\tau_2 = \omega\tau$ for the BDA1 and BDA2, respectively, where

$$\omega = \frac{\tau_2}{\tau_1} = \frac{\mu\nu}{(1-\mu)(1-\nu)}. \qquad (9)$$

Note that $\omega = 1$ in the ideal case for $\mu+\nu = 1$ and $\omega = 0.726$ for the applied PDBS. Second, the realization of the MPCC with the PDBS where $\mu + \nu \neq 1$ requires applying an additional unconditional filtering. This filtering is polarization-dependent and is performed regardless of the state of the ancillary photon. The polarization dependent transmittances τ_H and τ_V for the H and V-polarized photons, respectively, need to satisfy the following relation:

$$\kappa = \frac{\tau_V}{\tau_H} = \frac{2\mu - 1}{1 - 2\nu}, \qquad (10)$$

where κ ($\kappa = 1$ for $\mu + \nu = 1$) is a constant value fixed by the parameters of the PDBS (in our case $\kappa = 0.838$), and both τ_H and τ_V should have the largest possible values in order to maximize the efficiency of the setup. Therefore, we apply an additional unconditional filtering only for the V-polarized photons since the optimal transmittances are $\tau_V = \kappa$ and $\tau_H = 1$. Please note that for our PDBS $\kappa < 1$ and in the opposite case the best choice of the parameters is $\tau_V = 1$ and $\tau_H = 1/\kappa$. Moreover, if there are any other systematic uniform polarization-dependent losses τ'_H and τ'_V we can compensate for them by setting $\kappa = \tau'_H/\tau'_V \times (2\mu - 1)/(1 - 2\nu)$.

To summarize the above-mentioned corrections, the overall filtering operations in the first mode are described by

$$\tau_{1,H} = \tau^{\delta_{V,s}} \text{ and } \tau_{1,V} = \kappa\tau^{\delta_{H,s}}, \qquad (11)$$

and in the second mode by

$$\tau_{2,H} = (\omega\tau)^{\delta_{V,s}} \text{ and } \tau_{2,V} = \kappa(\omega\tau)^{\delta_{H,s}}, \qquad (12)$$

where $\delta_{V,s}$ ($\delta_{H,s}$) is Kronecker's delta and is equal to 1 iff the polarization s of the ancillary photon is V (H).

To implement the required filtering, additional polarization-independent filters FA1 and FA2 are placed at the output modes. These two filters together with the filters in both BDAs are sufficient to perform filtering operation described by Eqs. (11) and (12).

The usage of additional filtering saves the maximum achievable fidelity at the expense of lowering the success probability of the scheme. Using PDBA transmittances and the parameter Λ of the cloned state one can express the expected success probability of the scheme in the form of

$$P_{\mathrm{th}} = (1-2\mu)^2/2 + \mu\nu\tau\kappa, \qquad (13)$$

where $\kappa = (2\mu - 1)/(1 - 2\nu)$.

IMPLEMENTING THE GENERALIZED PCC AND AXISYMMETRIC CLONING

By using the same setup we implemented the generalized PCC (see Tab. I) which is a special case of the axisymmetric cloner described in Ref. [3]. In order to perform arbitrary axisymmetric cloning we set parameters of filters according to the following relations:

$$\begin{cases} \frac{\tau_{1,H}}{\tau_{1,V}} = \left(\frac{\cos\alpha_+}{\sin\alpha_-}\right)^2 \frac{2(1-\mu)(1-\nu)}{(1-2\mu)^2} \\ \frac{\tau_{2,H}}{\tau_{2,V}} = \left(\frac{\cos\alpha_+}{\sin\alpha_-}\right)^2 \frac{2\mu\nu}{(1-2\mu)^2} \end{cases} \quad \text{for } s = H, \qquad (14)$$

and

$$\begin{cases} \frac{\tau_{1,H}}{\tau_{1,V}} = \left(\frac{\sin\alpha_+}{\cos\alpha_-}\right)^2 \frac{(2\nu-1)^2}{2(1-\mu)(1-\nu)} \\ \frac{\tau_{2,H}}{\tau_{2,V}} = \left(\frac{\sin\alpha_+}{\cos\alpha_-}\right)^2 \frac{(2\nu-1)^2}{2\mu\nu} \end{cases} \quad \text{for } s = V, \qquad (15)$$

where $\alpha_\pm$ is given by Eq. (14) from [3] and $s = H, V$ stands for polarization of the ancillary state. For the generalized PCC we picked s deterministically. We set $\alpha_+ = \pi/2$ and $\alpha_- = 0$ for $s = H$ when we cloned a qubit from the northern hemisphere, alternatively we set $\alpha_+ = 0$ and $\alpha_- = \pi/2$ for $s = V$ each time when the cloned qubit was from the southern hemisphere. Please note that we could have also picked ancillary state at random (as for the MPCC), but then we would have had to block all the output modes for half of the cases.

MEASURING THE SUCCESS PROBABILITY

In order to measure success probability we need to estimate the inherent technological losses of the scheme and the initial photonic rate. The technological losses occur as a result of detector efficiencies, fiber coupling losses or back reflections. The coincidence rate $C_{\rm clon}$ measured at the end of the working cloner can be expressed as

$$C_{\rm clon} = P_{\rm ex}\tau_{\rm tech}C_{\rm init}, \tag{16}$$

where $P_{\rm ex}$ denotes the success probability of the cloning scheme, $\tau_{\rm tech}$ denotes the transmittance of the setup due to technological losses and $C_{\rm init}$ is the initial rate of photon pairs from the source. To compensate for the technological losses and the initial photon rate we use the following calibration procedure: PDBS is placed on a translation state allowing us to shift it slightly so that the reflected beam is no longer coupled. We use $|H_1H_2\rangle$ for the input state knowing that the beam splitter would decrease the coincidence rate by the factor of $1/\mu^2$. In this configuration we remove all the neutral density filters and measure the calibration coincidence rate $C_{\rm calib}$ at the end of the scheme. One can clearly see that

$$C_{\rm calib} = \mu^2\tau_{\rm tech}C_{\rm init} \tag{17}$$

so the success probability of the cloning operation can be expressed by combining Eqs. (16) and (17):

$$P_{\rm ex} = \mu^2\frac{C_{\rm clon}}{C_{\rm calib}}. \tag{18}$$

This equation allows us to obtain the success probability of the cloning operation from the measurement of two coincidence rates: the first is the coincidence rate of the working cloner and the second is the calibration coincidence rate. Note that Eqs. (13) and (18) describe the same quantity.

MEASURED VALUES

Our detailed summary of measured and predicted results is presented in Tables I, II, and III. In Fig. 1 we show how the cloning fidelity of the MPCC varies with phase φ.

Angle θ	$F_{\rm ex}$ [%]	$F_{\rm th}$ [%]
0	99.8 ± 0.4	100.0
$\pi/12$	99.3 ± 0.4	99.8
$\pi/5$	98.0 ± 0.8	98.8
$\pi/3$	95.7 ± 0.8	95.3
$3\pi/8$	92.4 ± 1.5	93.4
$\pi/2.25$	88.7 ± 1.1	89.4
$\pi/2$	84.1 ± 0.5	85.4
$\pi/1.8$	87.9 ± 0.7	89.4
$5\pi/8$	91.3 ± 1.0	93.4
$2\pi/3$	95.0 ± 0.8	95.3
$4\pi/5$	97.9 ± 0.7	98.8
$11\pi/12$	98.4 ± 1.0	99.8
π	99.8 ± 0.4	100.0

TABLE I: Summarized data for the PCC regime. $F_{\rm ex}$ denotes experimentally estimated average fidelity for a given polar angle θ on the Bloch sphere and $F_{\rm th}$ is the theoretical prediction. Note that the error estimated as RMS is just indicative of the actual error, because it does not take into account the physical properties of fidelity.

Angle θ	$F_{\rm ex}$ [%]	$F_{\rm th}$ [%]	$P_{\rm ex}$ [%]	$P_{\rm th}$ [%]
0	99.6 ± 0.4	100.0	10.5 ± 2.8	13.3
$\pi/12$	95.6 ± 1.7	97.0	10.6 ± 1.9	13.3
0.43	89.6 ± 0.4	90.4	10.4 ± 1.5	13.5
$\pi/5$	86.1 ± 1.6	87.4	9.6 ± 0.9	14.3
$\pi/4$	81.9 ± 2.0	84.1	14.0 ± 2.9	16.2
0.95	80.2 ± 1.5	83.3	19.5 ± 3.5	18.6
$3\pi/8$	82.3 ± 1.3	84.0	23.7 ± 1.5	21.7
$\pi/2$	84.1 ± 0.5	85.4	24.8 ± 0.1	24.0
$5\pi/8$	82.3 ± 1.3	84.0	23.7 ± 1.5	21.7
$2\pi/3$	80.2 ± 1.5	83.3	19.5 ± 3.5	18.6
2.19	81.9 ± 2.0	84.1	14.0 ± 2.9	16.2
$4\pi/5$	86.1 ± 1.6	87.4	9.6 ± 0.9	14.3
2.71	89.6 ± 0.4	90.4	10.4 ± 1.5	13.5
$11\pi/12$	95.6 ± 1.7	97.0	10.6 ± 1.9	13.3
π	99.6 ± 0.4	100.0	10.5 ± 2.8	13.3

TABLE II: Same as in Table I but for the MPCC regime. Moreover $P_{\rm ex}$ and $P_{\rm th}$ denote experimental and theoretical success probabilities.

THE MPCC AND PAULI DAMPING CHANNEL

Here, we show that our implementation of the MPCC can be interpreted as a quantum simulation of the Pauli dampening channel, where an error (bit-flip error, phase-flip error or both) occurs with some probability. This correspondence can lead to immediate applications of the proposed device for quantum eavesdropping. The density matrices of both clones are the same as the density matrix of the copied state transmitted via

Polarization state	$F_{\rm ex}$ [%]	$F_{\rm th}$ [%]
horizontal	80.2 ± 3.1	83.3
diagonal	81.5 ± 1.5	83.3
anti-diagonal	81.3 ± 0.2	83.3
right-circular	82.5 ± 1.4	83.3
left-circular	80.1 ± 0.9	83.3
vertical	83.2 ± 0.3	83.3

TABLE III: Same as in Tables I and II but for the UC regime and various polarization states.

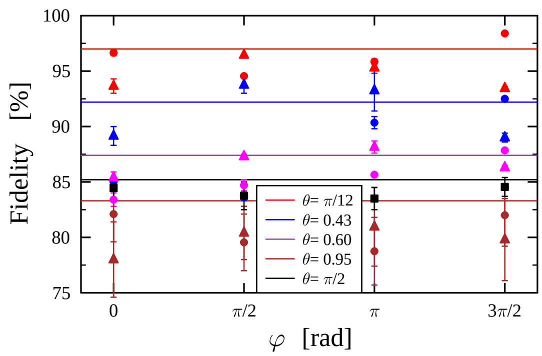

FIG. 1: Phase (angle φ) dependence of fidelity of the MPCC for the selected values (see Table II) of angle θ.

the noisy channel,

$$\hat{\rho}_{\rm out} = \alpha_+ \hat{\rho}_{\rm in} + \frac{\bar{\Lambda}^2}{4}(\hat{\sigma}_x\hat{\rho}_{\rm in}\hat{\sigma}_x + \hat{\sigma}_y\hat{\rho}_{\rm in}\hat{\sigma}_y) + \alpha_- \hat{\sigma}_z\hat{\rho}_{\rm in}\hat{\sigma}_z,$$

where $\alpha_\pm = \left(1+\Lambda^2 \pm 2\sqrt{2}\Lambda\bar{\Lambda}\right)/4$ and $\Lambda^2 + \bar{\Lambda}^2 = 1$. The parameter Λ depends on the distribution g of the cloned qubits and $\hat{\rho}_{\rm in} = |\psi\rangle\langle\psi|$. In the special case for $\Lambda^2 = 2/3$, the channel becomes so-called depolarizing channel, where the probability of all errors is the same and equal to 1/12. In such case the corresponding cloning machine is the UC. Moreover, for $\Lambda^2 = 1$ the channel becomes a dephasing channel (only the phase-flip error can occur) and the corresponding cloning is optimized for covariant cloning of the eigenstates of the phase-flip operator $|0\rangle$ and $|1\rangle$ and, thus, those two states can be perfectly copied or transmitted through the lossy channel.

* Electronic address: lemr@jointlab.upol.cz

† Electronic address: karol.bartkiewicz@amu.edu.pl

[1] A. Jamiołkowski, Rep. Math. Phys. **3**, 275 (1972).

[2] K. Audenaert and B. De Moor, Phys. Rev. A **65**, 030302 (2002).

[3] K. Bartkiewicz and A. Miranowicz, Phys. Rev. A **82**, 042330 (2010).