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Optimal cloning of arbitrary mirror-symmetric distributions on the Bloch sphere: a proposal for practical photonic realization

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Abstract
We study state-dependent quantum cloning that can outperform universal cloning (UC). This is possible by using some *a priori* information on a given quantum state to be cloned. Specifically, we propose a generalization and optical implementation of quantum optimal mirror phase-covariant cloning, which refers to optimal cloning of sets of qubits of known modulus of the expectation value of Pauli's Z operator. Our results can be applied to cloning of an arbitrary mirror-symmetric distribution of qubits on the Bloch sphere including in special cases UC and phase-covariant cloning. We show that the cloning is optimal by adapting our former optimality proof for axisymmetric cloning (Bartkiewicz and Miranowicz 2010 *Phys. Rev.* A **82** 042330). Moreover, we propose an optical realization of the optimal mirror phase-covariant $1 \rightarrow 2$ cloning of a qubit, for which the mean probability of successful cloning varies from $1/6$ to $1/3$ depending on prior information on the set of qubits to be cloned. The qubits are represented by polarization states of photons generated by the type-I spontaneous parametric down-conversion. The scheme is based on the interference of two photons on an unbalanced polarization-dependent beam splitter with different splitting ratios for vertical and horizontal polarization components and the additional application of feedforward by means of Pockels cells. The experimental feasibility of the proposed setup is carefully studied including various kinds of imperfections and losses. Moreover, we briefly describe two possible cryptographic applications of the optimal mirror phase-covariant cloning corresponding to state discrimination (or estimation) and secure quantum teleportation.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The no-cloning theorem [1, 2] states that unknown quantum states cannot be copied perfectly, which is implied by the linearity of quantum mechanics. The no-cloning theorem guarantees, e.g., the security (or privacy) of quantum communication protocols including quantum key distribution and excludes naive protocols of superluminal communication with entangled particles.

As perfect quantum cloning is impossible, much attention has been devoted to approximate [2–4] and probabilistic [5] quantum cloning. Such studies are particularly important not only for quantum cryptography [6, 7] and quantum state estimation [8], but also for quantum communication [9] and quantum computation [10].

It is worth noting that quantum cloning is not only of theoretical interest. In fact, a few experimental realizations of quantum cloning have been reported [11–15]. In particular,

quantum cloning with prior partial information, which is the main topic of our paper, was experimentally demonstrated using nuclear magnetic resonance [16, 17] and optical systems [18–23]. Also quantum-dot implementations of cloning machines were considered [24, 25].

The first $1 \to 2$ optimal cloning machine was designed by Bužek and Hillery [3]. This cloning machine, referred to as the universal cloning machine (UC), prepares two approximate copies of an unknown pure qubit state with the same fidelity $F = 5/6$. This means that the UC is state independent (i.e. the cloning is equally good for any pure qubit state) and symmetric (i.e. the copies are identical). The case of the UC producing an infinite number of copies [26] allowed the classical limit of $F = 5/6$ for copying quantum information to be established, which corresponds to the best copying operation achieved by classical operations.

The concept of optimal cloning was further extended to include cloning of qudits, cloning of continuous-variable systems and state-dependent cloning (non-universal cloning). The state-dependent cloning machines can produce clones of a specific set of qubits with higher fidelity than that for the UC, i.e. $F = 5/6$ [27–37] (see also the reviews [38, 39] and references therein). The study of state-dependent cloning is well motivated since we often have some *a priori* information about a given quantum state that we want to clone and by employing the available information, we can construct a cloning machine that surpasses the UC for some *a priori* specified set of qubits. For example, if the qubits are taken from the equator of the Bloch sphere, then by using the so-called optimal *phase-covariant cloners* (PCCs) [30, 34], one can achieve the fidelity $F = 1/2(1 + 1/\sqrt{2})$ much higher than $F = 5/6$.

The phase-covariant and phase-independent clonings were further generalized by Fiurášek [32], who studied the PCCs of qubits of known expectation value of Pauli's $Z \equiv \hat{\sigma}_z$ operator and provided two optimal symmetric cloners: one for the states in the northern hemisphere and the other for those in the southern hemisphere of the Bloch sphere.

Further works on phase-independent cloning described cloning of qubits uniformly distributed on a belt of the Bloch sphere [35] and the so-called *mirror phase-covariant cloning* (MPCC) [36], for qubits of known modulus of the expectation value of Pauli's $\hat{\sigma}_z$ operator. Finally, the unified approach to $1 \to 2$ cloning of arbitrary phase-independent distributions of qubits was presented in [37].

In this paper, we propose an optical implementation of the MPCC [36] based on a generalized version of the setup described by Černoch *et al* [18] (see also [20]). The experimental setup can equally well perform operations of the UC, PCC and MPCC in special cases corresponding to the proper choice of Λ (the explicit formulae can be found in section 2).

In the following sections, we analyze the performance of our setup accounting for various losses and imperfections such as finite efficiency of generating a pair of entangled photons in the type-I spontaneous parametric down conversion (SPDC), the influence of choosing various parameters of an unbalanced beam splitter (splitting ratios for vertical and horizontal polarization components), finite detector efficiency, dark counts and finite resolution of applied detectors. For simplicity, we neglect the effects of mode mismatch on the fidelity of the MPCC. Analysis of such losses would require application of a pulse-mode formalism (see, e.g., [42]).

Our paper is organized as follows. In section 2, we describe the standard MPCC and introduce a generalized MPCC. In section 3, we present a setup implementing the optimal symmetric $1 \to 2$ MPCC of a qubit and study the influence of imperfections of the beam splitter on the performance of this cloning. In section 4, we study the performance of the setup assuming imperfect photon detectors by means of the positive operator valued measure (POVM) formalism. The applicability of the proposed setup to implement the generalized MPCC is shown in section 5. We conclude in section 6.

2. Mirror phase-covariant cloning

The cloning transformation for the MPCC requires one ancilla and has the following unitary form in the computational basis:

$$\begin{aligned} |0\rangle_{\rm in} &\to \Lambda|00\rangle_{1,2}|0\rangle_{\rm anc} + \bar{\Lambda}|\psi_+\rangle_{1,2}|1\rangle_{\rm anc}, \\ |1\rangle_{\rm in} &\to \Lambda|11\rangle_{1,2}|1\rangle_{\rm anc} + \bar{\Lambda}|\psi_+\rangle_{1,2}|0\rangle_{\rm anc}, \end{aligned} \tag{1}$$

where $\Lambda^2 + \bar{\Lambda}^2 = 1$ and $|\psi_+\rangle = 1/\sqrt{2}(|01\rangle + |10\rangle)$ is one of the Bell states. The resulting clones are found in modes 1 and 2. The parameter Λ explicitly depends on the modulus of the expectation value of $\hat{\sigma}_z$ as follows:

$$\Lambda = \sqrt{\frac{1}{2} + \frac{\cos^2\theta}{2\sqrt{P}}}, \tag{2}$$

where $P \equiv P(\theta) = 2 - 4\cos^2\theta + 3\cos^4\theta$ and $\langle\hat{\sigma}_z\rangle = \cos\theta$. The average fidelity F over the Bloch sphere (note that MPCC is state dependent) is $F = 0.8594$ and is larger than the fidelity for the case of the UC, which is $F = 0.8333$.

Moreover, from [37] it follows that any optimal cloning machine that copies a phase-covariant set of qubits and exhibits mirror xy-plane symmetry is described by the same general transformation, where Λ depends on the set of qubits for which the cloning machine is optimized (see section 5).

Here, this optimal cloning of an arbitrary mirror-symmetric (and axisymmetric) distribution of qubits on the Bloch sphere will be referred to as the generalized MPCC. Therefore, the proposed experimental setup can be used for cloning an arbitrary set of qubits of the described symmetry. It is worth noting that former proposals of realizations of the MPCC in linear-optical systems [36, 37] and quantum dots [36, 40] were discussed formally without referring to experimental setups.

A question arises about the usefulness of the MPCC for quantum information or quantum-state engineering. Here, we briefly describe two possible cryptographic applications. Namely, we suggest using the MPCC for secure teleportation. It was discussed in [41] that states that are symmetric about the equator plane of Bloch's sphere enable secure teleportation. Now, the limits of secure teleportation can be studied by applying our optimal cloning transformation which can be used by an eavesdropper Eve. The axisymmetric cloners (ASC), corresponding to cloning of arbitrary

axisymmetric distributions, require more prior knowledge about qubit states to be teleported, while the optimal set of such states just corresponds to the MPCC. In other words, it is more practical to use the MPCC rather than the ASC in this case. As an another obvious application of the MPCC, we note that it can be used for effective estimation or discrimination of mirror-symmetric states.

3. A proposal for practical photonic implementation of mirror phase-covariant cloning

Here we present the main result of our paper which is an optical implementation of the optimal $1 \to 2$ MPCC of a qubit. In section 5, we show that the generalized MPCC and other known optimal cloning machines can be implemented by our setup.

3.1. Initialization

The initial entangled state is prepared by using parametric down conversion of the first type (see [43–46]). The output of a pulsed laser (PL), with angular frequency ω_0, is frequency doubled in a nonlinear crystal to produce pulses of ultraviolet (UV) light of angular frequency $2\omega_0$. The UV pulses are then used to pump twice (in the forward and backward directions) a pair of nonlinear crystals which are stacked together such that their optical axes are orthogonal to each other [43, 47]. The crystals are type-I SPDC to produce photon pairs in two modes (idler and signal) of the same polarization and of half the frequency of the PL. In the forward pumping direction, the polarization of the UV beam is set to vertical so that an H-polarized photon pair in modes 2 and $0'$ is generated. The remaining (not down-converted) portion of the UV beam first passes through a quarter-wave plate (QWP_1) which changes its polarization into an ellipsoidal polarization. A mirror M_1 placed after the QWP_1 reflects this beam and sends it through the QWP_1 again which further changes the polarization of the beam into diagonal polarization. This diagonally polarized beam pumps the crystals in the backward direction, creating the entangled photon pair $|\psi_+\rangle = (|1_\mathrm{H}\rangle_0|1_\mathrm{V}\rangle_1 + |1_\mathrm{V}\rangle_0|1_\mathrm{H}\rangle_1)/\sqrt{2}$. However, the total state of the system in modes 0, $0'$, 1 and 2 is more complex than

$$|\Psi\rangle = |\psi_+\rangle_{0,1}|1_\mathrm{H}\rangle_{0'}|1_\mathrm{H}\rangle_2, \tag{3}$$

which we use in our further analytical considerations. On the one hand, the SPDC is probabilistic and the state $|\Psi\rangle$ consists also of the vacuum and higher-order SPDC terms. On the other hand, for the circuit to work we require fourfold coincidence count in all modes and a very low dark-count rate of modern photon detectors (dark count probability of the order of 10^{-6}) allows us to effectively eliminate the vacuum state from the further considerations. The measurement of a photon in mode 0 is polarization dependent, which is further used in the feedforward processing. So, finally, the system is prepared in the state

$$|\Psi\rangle = \mathcal{N}\left[\gamma^2 \mathrm{e}^{2\mathrm{i}\phi}|\psi_+\rangle_{01}|1_\mathrm{H}\rangle_{0'}|1_\mathrm{H}\rangle_2 + \gamma^3 \mathrm{e}^{3\mathrm{i}\phi} \times \left(|\psi_+\rangle_{01}|2_\mathrm{H}\rangle_{0'}|2_\mathrm{H}\rangle_2 + |\epsilon\rangle_{01}|1_\mathrm{H}\rangle_{0'}|1_\mathrm{H}\rangle_2\right) + \mathcal{O}(\gamma^4)\right], \tag{4}$$

where

$$|\epsilon\rangle_{01} = \tfrac{1}{2}(|1_\mathrm{H}1_\mathrm{V}\rangle_0 + |1_\mathrm{H}1_\mathrm{V}\rangle_1 + |2_\mathrm{H}\rangle_0|2_\mathrm{V}\rangle_1 + |2_\mathrm{V}\rangle_0|2_\mathrm{H}\rangle_1).$$

Moreover, γ describes the efficiency of the SPDC and depends on the amplitude of the incident field and properties of the nonlinear crystal, ϕ is the phase shift caused by the SPDC and $\mathcal{N}$ is the normalization constant. Typically $\gamma^2 = 0.01$ [47], so for simplicity, we will almost always neglect the terms of amplitude of order higher than γ^2 since the probability of occurrence of such events is very low as $|\mathcal{O}(\gamma^3)|^2 = \mathcal{O}(\gamma^6)$. We will consider a more complete form of $|\Psi\rangle$ only in section 4.

Next, we prepare the arbitrary state to be cloned in mode 2 by a combination of a half-wave plate (HWP_1) and QWP_2. The input state is passed into mode 2. This is given in the following form:

$$|\psi\rangle_2 = (\alpha\hat{a}^\dagger_{2\mathrm{H}} + \beta\hat{a}^\dagger_{2\mathrm{V}})|0\rangle_2, \tag{5}$$

where $\alpha = \cos(\theta/2)$ and $\beta = \mathrm{e}^{\mathrm{i}\delta}\sin(\theta/2)$. Later, modes 1 and 2 are mixed on an unbalanced polarization-dependent beam splitter (PDBS). The PDBS transforms the input in the following way:

$$\begin{aligned}
\hat{a}^\dagger_{1\mathrm{H}} &\to \sqrt{1-\mu}\hat{a}^\dagger_{1'\mathrm{H}} - \sqrt{\mu}\hat{a}^\dagger_{2'\mathrm{H}},\\
\hat{a}^\dagger_{1\mathrm{V}} &\to \sqrt{1-\nu}\hat{a}^\dagger_{1'\mathrm{V}} + \sqrt{\nu}\hat{a}^\dagger_{2'\mathrm{V}},\\
\hat{a}^\dagger_{2\mathrm{H}} &\to \sqrt{\mu}\hat{a}^\dagger_{1'\mathrm{H}} + \sqrt{1-\mu}\hat{a}^\dagger_{2'\mathrm{H}},\\
\hat{a}^\dagger_{2\mathrm{V}} &\to \sqrt{\nu}\hat{a}^\dagger_{1'\mathrm{V}} - \sqrt{1-\nu}\hat{a}^\dagger_{2'\mathrm{V}}.
\end{aligned} \tag{6}$$

The MPCC can be implemented when $\mu + \nu = 1$. In fact, if the above condition is not fulfilled some compensation method can be applied ensuring maximal theoretical fidelity at the expense of the lower success probability of the setup. The most convenient situation is when $\mu = \mu_0 = (1 - 1/\sqrt{3})/2$ and $\nu = \nu_0 = (1 + 1/\sqrt{3})/2$, i.e. $1 - 2\mu = 2\nu - 1 = \sqrt{2\mu\nu} = 1/\sqrt{3}$. Analogous conditions for the PCC were given by Fiurášek [32]. Finally, the state of the system after the action of the PDBS (for $\mu = \nu$) is given by the following expression:

$$\begin{aligned}
|\Psi'\rangle = \mathcal{N}'[&\alpha\hat{a}^\dagger_{0\mathrm{V}}(x\hat{a}^\dagger_{1'\mathrm{H}}\hat{a}^\dagger_{1'\mathrm{H}} - x\hat{a}^\dagger_{2'\mathrm{H}}\hat{a}^\dagger_{2'\mathrm{H}} + y\hat{a}^\dagger_{1'\mathrm{H}}\hat{a}^\dagger_{2'\mathrm{H}})\\
&+ \beta\hat{a}^\dagger_{0\mathrm{V}}(\nu\hat{a}^\dagger_{1'\mathrm{H}}\hat{a}^\dagger_{1'\mathrm{V}} + \mu\hat{a}^\dagger_{2'\mathrm{H}}\hat{a}^\dagger_{2'\mathrm{V}} - x\hat{a}^\dagger_{1'\mathrm{H}}\hat{a}^\dagger_{2'\mathrm{V}} - x\hat{a}^\dagger_{1'\mathrm{V}}\hat{a}^\dagger_{2'\mathrm{H}})\\
&+ \alpha\hat{a}^\dagger_{0\mathrm{H}}(\mu\hat{a}^\dagger_{1'\mathrm{H}}\hat{a}^\dagger_{1'\mathrm{V}} + \nu\hat{a}^\dagger_{2'\mathrm{H}}\hat{a}^\dagger_{2'\mathrm{V}} + x\hat{a}^\dagger_{1'\mathrm{V}}\hat{a}^\dagger_{2'\mathrm{H}} + x\hat{a}^\dagger_{1'\mathrm{H}}\hat{a}^\dagger_{2'\mathrm{V}})\\
&+ \beta\hat{a}^\dagger_{0\mathrm{H}}(x\hat{a}^\dagger_{1'\mathrm{V}}\hat{a}^\dagger_{1'\mathrm{V}} - x\hat{a}^\dagger_{2'\mathrm{V}}\hat{a}^\dagger_{2'\mathrm{V}} + y\hat{a}^\dagger_{1'\mathrm{V}}\hat{a}^\dagger_{2'\mathrm{V}})]\hat{a}^\dagger_{2'\mathrm{H}}|0\rangle,
\end{aligned}$$

where $x = \sqrt{\mu\nu}$, $y = 1 - 2\mu$, $|0\rangle = |0\rangle_{00'1'2'}$ and $\mathcal{N}'$ is a normalization constant.

3.2. Feedforward

In order to implement the MPCC, we also apply a feedforward technique (see [48, 49]), i.e. photons of the same polarization as detected in mode 0 are damped in modes $1'$ and $2'$. The element implementing the damping is based on a Pockels cell and two PBSs and is presented in figure 1. As was shown in [48], such an operation can be performed with a high

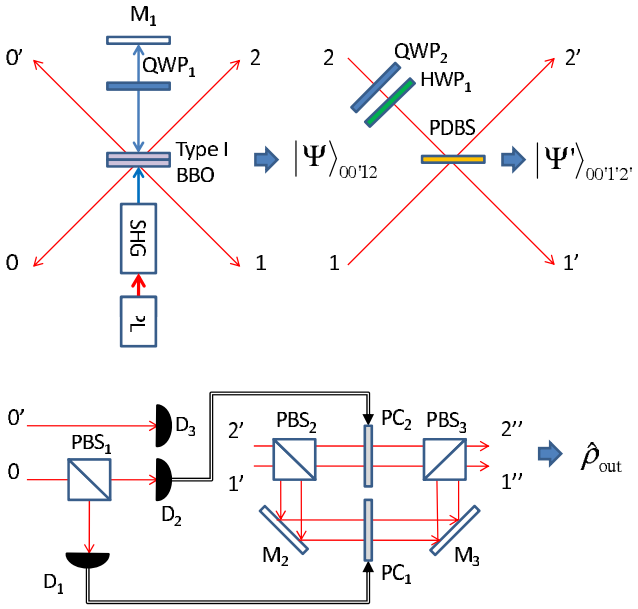

Figure 1. Scheme of the experimental setup used to implement the MPCC. We use the following acronyms for the standard optical elements: QWP, quarter-wave plate; HWP, half-wave plate; PBS, balanced polarizing beam splitter; PDBS, polarization-dependent beam splitter for different splitting ratios for H and V polarizations; D, detector; PC, Pockels cell; PL, pulsed laser; SHG, second-harmonic generation; and M, mirror. Double solid lines denote transfer of classical information. The intermediate state $|\Psi\rangle_{00'12}$ is prepared by means of SPDC of type-I (see, e.g., [43]) using a stack of two β-barium borate (BBO) crystals. Next, $|\Psi\rangle_{00'12}$ is transformed into $|\Psi'\rangle_{0,0'1'2'}$ by first setting the input state $|\psi\rangle_2$ to be cloned with the QWP_2 and HWP_1 and then mixing modes 1 and 2 on the PDBS (this can be considered as the first step of the actual cloning). Finally, $|\Psi'\rangle_{00'1'2'}$ is subject to classical feedforward, driven by the measurement outcomes of D_1 and D_2. As a result, we obtain state $\hat{\rho}_{\rm out}$ that is the outcome of the cloning machine as long as there is one photon in modes $1''$ and $2''$. Detector D_3 is used as a trigger for the experiment, which practically eliminates the probability of having vacuum in mode 2. This is due to the low dark-count rate of modern photon detectors.

fidelity of more than 99%. The final density matrix of the system is given as

$$\hat{\rho}_{\rm out}=\mathrm{Tr}_{00'}\left[\left(\hat{\Pi}^0_{1\mathrm{H}}\hat{\Pi}^0_{0\mathrm{V}}\hat{D}_{\mathrm{H}}\hat{\rho}'\hat{D}^\dagger_{\mathrm{H}}+\hat{\Pi}^0_{1\mathrm{V}}\hat{\Pi}^0_{0\mathrm{H}}\hat{D}_{\mathrm{V}}\hat{\rho}'\hat{D}^\dagger_{\mathrm{V}}\right)\hat{\Pi}^{0'}_1\right], \tag{7}$$

where $\hat{\rho}'=|\psi'\rangle\langle\psi'|$ is the output of the state after the action of the unbalanced PDBS, $\hat{D}_{\mathrm{H}}=\hat{\Gamma}_{1'\mathrm{H}}\hat{\Gamma}_{2'\mathrm{H}}$, $\hat{D}_{\mathrm{V}}=\hat{\Gamma}_{1'\mathrm{V}}\hat{\Gamma}_{2'\mathrm{V}}$, where $\hat{\Gamma}_{i\mathrm{V}}$ ($\hat{\Gamma}_{i\mathrm{H}}$) is the operation acting on photons in the ith spatial mode, which corresponds to the conditional application of a Pockels cell (see figure 1). $\hat{\Pi}^i_{j\mathrm{H}}$ ($\hat{\Pi}^i_{j\mathrm{V}}$) are the POVM operators describing the probability of detection of the j H(V)-polarized photons in the ith mode. The damping operation can be described as

$$\hat{\Gamma}_{i,s}|m_{\mathrm{H}},n_{\mathrm{V}}\rangle_j=[\delta_{ij}(-\lambda)^{n\delta_{s,\mathrm{V}}}\lambda^{m\delta_{s,\mathrm{H}}}+\tilde{\delta}_{ij}]|m_{\mathrm{H}},n_{\mathrm{V}}\rangle_j,$$

where $\delta_{i,j}$ is Kronecker's delta, $\tilde{\delta}_{ij}=1-\delta_{ij}$, i and j enumerate the spatial modes $1'$ and $2'$, while $\delta_{s,r}$ is Kronecker's delta, where s and r denote polarization modes H or V. Moreover, λ is a damping parameter, which in the case of a perfect PDBS is equal to $\lambda=\bar{\Lambda}/\Lambda$. Negative λ means that, in addition to damping, a phase flip needs to be applied.

3.3. Post-selection

The cloning procedure is successful as long as there is only one photon in every outgoing mode. The probability of the coincidence count (i.e. the probability of success) is given by the following expression:

$$P_{\rm success}=\mathrm{Tr}_{3,4}(\hat{\rho}_{\rm out}\hat{\Pi}^{1''}_1\hat{\Pi}^{2''}_1)=\frac{1}{6\Lambda^2}, \tag{8}$$

where $1/\sqrt{2}\leqslant\Lambda\leqslant 1$ for the MPCC. Hence, the probability of successful cloning $P_{\rm success}$ varies from 1/3 to 1/6 (given that we work with perfect detectors and a perfect source of entangled photons) depending on the states we want to clone in an optimal way. The proposed implementation is probabilistic, but the probability of successful cloning is much higher than in the case of using the simple quantum circuit proposed in [36], which requires four controlled NOT (CNOT) gates, with the best known nondestructive optical CNOT gates having a success rate of 1/4 due to Pittman [50] (for a review see [51]). The optimal cloner constructed in such a way will have a success rate of 1/256.

3.4. Fidelity of the proposed experimental setup

In order to describe the quality of the cloning we use single-copy fidelity

$$F_i=(\mathrm{Tr}\,\hat{\rho}_{\rm out})^{-1}\langle\psi|\mathrm{Tr}_{3-i}\hat{\rho}_{\rm out}|\psi\rangle. \tag{9}$$

However, to describe the overall performance of a cloning machine it is more convenient to use the average single-copy fidelity

$$F=\frac{1}{2}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\vartheta\; g(\vartheta,\phi)[F_1(\vartheta,\phi)+F_2(\vartheta,\phi)], \tag{10}$$

which is an average over all possible input qubits defined by the distribution function $g(\vartheta,\phi)$. In the case of the MPCC, the g distribution in (10) is given by

$$g_\theta(\vartheta,\phi)=\frac{1}{4\pi}[\delta(\vartheta-\theta)+\delta(\vartheta+\theta-\pi)], \tag{11}$$

in terms of Dirac's δ-function. Moreover, we added subscript θ to indicate *a priori* knowledge about the input state.

One can easily check that the resulting expression for the average single-copy fidelity is the same as that for the MPCC [36] and is given by

$$F\equiv F_1=F_2=\frac{1+\Lambda^2}{2}-\frac{1}{2}\Lambda(\Lambda-\bar{\Lambda}\sqrt{2})\sin^2\theta. \tag{12}$$

From (12) it follows that the average cloning fidelity over all possible input states of the MPCC (over all θ—the average is over all possible circles and their mirror-symmetric counterparts) is $F=0.8594$.

Note that for simplicity of exposition, we focus here on the MPCC, which is the simplest nontrivial example of cloning of mirror-symmetric distributions $g(\theta)$ on the Bloch

sphere, where $g(\theta)$ is the sum of two Dirac δ-functions. However, in the case of other mirror-symmetric phase-covariant qubit distributions, we obtain different values of the average cloning fidelity and the success rate of the proposed experimental setup. For example, in the case of the UC we obtain $F = 0.8333$ (the average is over the whole Bloch sphere) and $F = 0.8536$ (the average over the equator of the Bloch sphere) in the case of the PCC. For the UC and PCC, we have $\Lambda = \sqrt{2/3}$ and $\Lambda = 1/\sqrt{2}$, respectively.

4. Practical considerations for experimental implementation

4.1. Choosing the parameters of an unbalanced polarization-dependent beam splitter

In order to perform the required quantum transformation, in some cases one needs to use a PDBS with some strictly chosen values of reflectance or transmittance (see [18, 47]). However, there are no perfect polarization beam splitters (see [18, 47]). In practice one can apply some mechanisms to compensate for the imperfections of the beam splitter (see [18]). In our case we use feedforward and obtain less strict requirements on optical realization of the symmetric covariant cloner than stated in [18], where $1-\nu = 1/2(1+1/\sqrt{3})$ and $1-\mu = 1/2(1-1/\sqrt{3})$ must have fixed values.

In the proposed experimental realization, it is enough to satisfy the condition $\mu+\nu = 1$. Otherwise, the single-copy fidelity drops and the cloning is no longer symmetric. Given that $\mu+\nu = 1$ is satisfied, it is enough that the damping parameter

$$\lambda = \frac{\bar{\Lambda}(1-2\mu)}{\Lambda\sqrt{2\mu\nu}} \tag{13}$$

and the fidelity of a single clone are the same as in the perfect case (please note that only $|\lambda| \leqslant 1$ is physical). Hence, imperfections of the PDBS (given that $\mu+\nu = 1$) result only in decreasing the success rate of the setup. Therefore, the probability of successful cloning is given by the following expression:

$$P_{\text{success}} = \frac{(1-2\mu)^2}{2\Lambda^2}, \tag{14}$$

where

$$\frac{1}{2}\left(1-\frac{1}{\sqrt{3}}\right) \leqslant \mu \leqslant \frac{1}{2}\left(1+\frac{1}{\sqrt{3}}\right). \tag{15}$$

We numerically compared the average over Bloch's sphere cloning fidelities of the first and second clones and its average over the two clones for the MPCC, which operation depends on $|\hat{\sigma}_z|$ of the cloned pure state. Our analysis shows that the average fidelity over the two clones reaches its maximum $F = 0.8594$ for $\nu = 1-\mu$ and μ given by (15). We observed that the cloning fidelity is symmetric for the parameters close to $\nu = 1-\mu$. The areas of high fidelity (> 0.85) are large in all three cases. Thus, the setup is robust to variations of μ or ν.

Moreover, we numerically studied the average success probability of the proposed setup in the case of the MPCC, which corresponds to the probability of finding one photon in both modes $1''$ and $2''$. The probability of successful coincidence count in both modes increases radially from the center of (μ, ν) space (balanced polarization-independent beam splitter) where it reaches zero. One can find the best μ and ν by finding such conditions for which the average fidelity and the probability of success are simultaneously maximized. This happens when $\nu = 1-\mu$ and the inequality given in (15) is saturated.

4.2. Influence of detector imperfections

Detectors play an important role in the proposed experiment. As one can see in (7), the density matrix $\hat{\rho}_{\text{out}}$ depends explicitly on the measurements carried out on the ancillary qubits. Also in practical realizations of the cloning machine, the fidelity of the cloning process can be evaluated by measuring the polarization of photons in modes $1''$ and $2''$ in the basis of $|\psi\rangle$ and $|\overline{\psi}\rangle$ as described in [18]. This gives seven detectors in total; however, we analyze only the cases when four detectors (one photon per detector) click at the same time. For simplicity, we assume that all the detectors are characterized by the same parameters.

There are two basic types of photon detectors that can be used in the experiment: single-photon counters and ON/OFF detectors. Since we cannot exclude completely the possibility of the higher-order SPDC events (see (4)), we investigate the implications of using both types of detectors.

4.2.1. Single-photon counters. First we analyze single-photon counters, which can discriminate between vacuum, detection of one photon and detection of many photons. We describe imperfections of these detectors by the following POVM operators [52, 53]:

$$\begin{aligned}\hat{\Pi}_0 &= \sum_{m=0}^{\infty} \mathrm{e}^{-\zeta}(1-\eta)^m |m\rangle\langle m|,\\ \hat{\Pi}_1 &= \sum_{n=0}^{1}\sum_{m=n}^{\infty} \mathrm{e}^{-\zeta}\zeta^{1-n}\eta^n m^n (1-\eta)^{m-n}|m\rangle\langle m|,\\ \hat{\Pi}_{N\geqslant 2} &= \hat{1} - \hat{\Pi}_0 - \hat{\Pi}_1,\end{aligned} \tag{16}$$

where η is quantum efficiency of the detectors and ζ stands for the dark-count rate (typically of the order of 10^{-6}).

4.2.2. ON/OFF detectors. We also analyze the ON/OFF detectors (also referred to as conventional or bucket detectors), which can discriminate only between vacuum and any other number of photons. The difference between the single-photon counters and ON/OFF detectors is negligible in the case of a low dark-count rate. Since we are interested only in such events where the number of detector 'clicks' is equal to the assumed number of photons in the system, we use the following POVM operators [52]:

$$\hat{\Pi}_0 = \sum_{m=0}^{\infty} \mathrm{e}^{-\zeta}(1-\eta)^m |m\rangle\langle m|, \quad \hat{\Pi}_{N\geqslant 1} = \hat{1} - \hat{\Pi}_0. \tag{17}$$

4.2.3. Expected fidelity and probability of cloning. In our proposed experimental setup, we use post-selection; thus the average fidelities of the clones can be expressed via coincidences as [18]

$$F_1 = \frac{C_{11}+C_{10}}{P_{\text{success}}}, \quad F_2 = \frac{C_{11}+C_{01}}{P_{\text{success}}}, \tag{18}$$

Table 1. The influence of imperfections of the detectors (neglecting dark counts) on the average success rate P_{success} and the average fidelities F_1 and F_2 of two clones, where η is the detector efficiency (some achievable values can be found in [54, 55]). The results show that the proposed cloning machine is essentially robust to finite efficiency and finite resolution of detectors. The loss of fidelity caused by imperfections is less than 1%. However, the theoretical limit of the maximal cloning fidelity can be reached only in the case of photon-number-discriminating detectors.

η	$P^{\text{a}}_{\text{success}}$	F^{a}_1	F^{a}_2	$P^{\text{b}}_{\text{success}}$	F^{b}_1	F^{b}_2
1.00	0.2552	0.8594	0.8594	0.2598	0.8567	0.8569
0.80	0.0671	0.8588	0.8589	0.0688	0.8555	0.8558
0.60	0.0120	0.8576	0.8578	0.0124	0.8540	0.8543
0.50	0.0041	0.8567	0.8569	0.0042	0.8531	0.8534
0.40	0.0011	0.8555	0.8558	0.0011	0.8521	0.8524

[a]Single-photon counters.
[b]ON/OFF detectors.

where

$$P_{\text{success}} = C_{00} + C_{01} + C_{10} + C_{11} \tag{19}$$

is the probability of success (i.e. the successful post-selection) and C_{ij} $(i, j \in \{0, 1\})$ are the following coincidences:

$$C_{11} = \text{Tr}\,(\hat{\rho}_{\text{out}}\hat{\Pi}_1 \otimes \hat{\Pi}_1), \quad C_{10} = \text{Tr}\,(\hat{\rho}_{\text{out}}\hat{\Pi}_1 \otimes \hat{\overline{\Pi}}_1),$$
$$C_{01} = \text{Tr}\,(\hat{\rho}_{\text{out}}\hat{\overline{\Pi}}_1 \otimes \hat{\Pi}_1), \quad C_{00} = \text{Tr}\,(\hat{\rho}_{\text{out}}\hat{\overline{\Pi}}_1 \otimes \hat{\overline{\Pi}}_1).$$

Here, the POVMs $\hat{\Pi}_1$ and $\hat{\overline{\Pi}}_1$ correspond to the detection of a photon in the states $|\psi\rangle$ and $|\overline{\psi}\rangle$, respectively. As one can see in (7), $\hat{\rho}_{\text{out}}$ depends on the quality and type of the photon detectors. Moreover, it also depends on the efficiency of generation of the entangled photon pairs (see (4)). In the case of perfect detectors (both single-photon counters and ON/OFF detectors) we have $\hat{\Pi}_1 = |1_\psi\rangle\langle 1_\psi|$. The influence of imperfections of measurements on the fidelity of cloning and success rate for single-photon counters (ON/OFF detectors) is summarized in tables 1 and 2. Note that our numerical results presented in table 1 indicate a surprising effect that the fidelity F decreases with decreasing detector efficiency η. One could predict that the smaller the η the lower the success probability P_{success}, but not necessarily the smaller the F. The loss of F is minor (in the third decimal place), but still cannot be treated as numerical noise. In fact, this effect can be explained by the generation of undesired states, proportional to γ^3 in (4), in the BBO crystals. These multiphoton states can be wrongly interpreted as the correct states if some of the photons are not detected assuming $\eta < 1$.

In summary, tables 1 and 2 show that our optical cloning machine is essentially robust to finite efficiency and finite resolution of detectors.

5. Applicability to arbitrary mirror-symmetric phase-covariant cloning

For simplicity, so far we have analyzed the setup for the MPCC [36] alone. However, the general cloning transformation given in (1) is optimal for the cloning of arbitrary mirror-symmetric distributions on the Bloch sphere. Here, this cloning machine is referred to as the generalized MPCC.

Table 2. The influence of finite dark-count rate of the detectors assuming perfect efficiency on the average success rate P_{success} and the average fidelities F_1 and F_2 of two clones, where ζ is the detector's dark-count rate. The influence of the dark counts for the usual dark-count rates (ζ of the order of 10^{-6} [52]) is negligible. For both types of detectors, the setup is robust (less than 1% drop of the average fidelity) up to ζ of the order of 0.001. It is seen that the probability of coincidence count increases with ζ for the ON/OFF detectors and drops in the case of the photon-number-discriminating detectors (single-photon counters). The ON/OFF detectors register false successful events as true coincidences. The single-photon counters are better in the case of low dark-count rates (most of the practical situations), but for $\zeta > 0.0001$ we observe that the performance of the machine is better when the ON/OFF detectors are applied.

ζ	$P^{\text{a}}_{\text{success}}$	F^{a}_1	F^{a}_2	$P^{\text{b}}_{\text{success}}$	F^{b}_1	F^{b}_2
10^{-6}	0.2552	0.8594	0.8594	0.2598	0.8567	0.8569
10^{-4}	0.2550	0.8589	0.8589	0.2598	0.8566	0.8568
10^{-2}	0.2403	0.8094	0.8094	0.2620	0.8470	0.8472

[a]Single-photon counters.
[b]ON/OFF detectors.

Recently, we showed [37] that the optimal symmetric $1 \to 2$ of an arbitrary axisymmetric qubit distribution $g(\theta)$, which is the distribution of expectation values $\langle\hat{\sigma}_z\rangle = \cos\theta$ for a set of qubits. We call these optimal cloning machines the axisymmetric cloners. Any $g(\theta)$ can be expanded in the basis of the Legendre polynomials $P_n(\cos\theta)$ [56] as

$$g(\theta) = \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)a_n P_n(\cos\theta), \tag{20}$$

$$a_n = \int_0^{2\pi}\int_{-1}^{1} g(\theta)\,P_n(\cos\theta)\,\mathrm{d}\cos\theta\,\mathrm{d}\phi. \tag{21}$$

In [37], we showed that the optimal cloning transformation depends only on the first three terms of this expansion. Moreover, we obtained $a_1 = 0$ for a normalized $(a_0 = 1)$ mirror-symmetric distribution, i.e. invariant to the action of the discrete Weyl–Heisenberg group. Such a case includes as special cases the PCC for $\theta = \pi/2$, the MPCC [36] and the UC of Bužek and Hillery [3].

By comparing the results from [36] with those from [37], we find that $\bar{\Lambda}$, given in (1), in general depends on a single parameter as follows:

$$\Lambda = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{8(1-a_2)^2}{3(3+4a_2^2-4a_2)}}}. \tag{22}$$

Thus, by using an appropriate functional form of Λ, we can implement various optimal cloning machines such as the PCC, MPCC and UC with the same experimental setup. Note that for the UC, $a_2 = 0$, and for the PCC, $a_2 = -1/2$, i.e. $\Lambda = \sqrt{2/3}$ and $\Lambda = 1/\sqrt{2}$, respectively.

6. Conclusions

We investigated experimentally feasible optimal mirror phase-covariant cloning, i.e. optimal cloning of arbitrary sets of qubits of known modulus of expectation value of Pauli's $\hat{\sigma}_z$ operator. Our definition of MPCC includes in special cases

the UC (corresponding to cloning of a uniform distribution of qubits on the Bloch sphere) and the phase-covariant cloning (cloning of equatorial qubits). By identifying the class of mirror-symmetric phase-covariant distributions of qubits as a subclass of axisymmetric distributions, for which the optimal cloning transformations were obtained in [37], we showed that the cloning transformation we implemented is optimal.

We briefly discussed two possible cryptographic applications of the MPCC corresponding to state discrimination (or estimation) and secure quantum teleportation.

We proposed an optical realization of the optimal quantum mirror phase-covariant $1 \to 2$ cloning of a qubit, for which the mean probability of successful cloning varies from $1/6$ to $1/3$ depending on prior information on the set of qubits to be cloned. The qubits are represented by polarization states of photons generated by SPDC of the first type. The scheme is based on the interference of two photons on a beam splitter with different splitting ratios for vertical and horizontal polarization components and additional application of feedforward by means of Pockels cells.

It could be argued that the feedforward procedure can be replaced by using a random ancilla state and the corresponding switching of filtering. This alternative method is simpler from a technical point of view, but this would be in fact a pseudo-random process since it is impossible to obtain a true random-number generator with deterministic devices. This property is of great importance for using this cloner for cryptographic purposes.

The phase-covariant cloning machine implemented by Černoch *et al* [18] is less general as it does not include feedforward, which allows the setup in cases other than implementation of the PCC to be used. Moreover, we showed that the feedforward also allows the use of splitting ratios of the PDBS of a wider range than that in the schemes without feedforward.

The experimental feasibility of the proposed setup was studied including various kinds of losses: (i) finite efficiency of generating a pair of entangled photons in the type-I SPDC, (ii) the influence of choosing various splitting ratios of an unbalanced beam splitter, (iii) the use of conventional (ON/OFF detectors) and single-photon discriminating detectors, (iv) finite efficiency of detectors and (v) their dark counts.

For simplicity, we studied the experimental feasibility of our setup implementing only the standard MPCC, i.e. which corresponds to cloning distribution $g(\theta)$ described by two Dirac δ-functions. Such an analysis can be easily extended to show the feasibility of our setup for the optimal cloning of arbitrary distributions $g(\theta)$ that are mirror symmetric on the Bloch sphere as described in section 5.

We showed that the cloning machine is robust to losses and imperfections; its fidelity is expected to be very close to the theoretical limit and is expected to stay unaffected by the imperfections of the particular elements other than the PDBS. Robustness of the proposed experimental setup was confirmed by an investigation of the influence of the mentioned imperfections on the average fidelity of clones and the success probability of the MPCC.

Both the success rate and average cloning fidelity were estimated by means of a simplified qubit tomography (see figure 2) setup [57]. In our case, similarly to Černoch *et al* [18], we do not need to use the complete tomography to determine the fidelity of the clones (since we *a priori* know the input state to some extent). The probability of successful cloning is high in comparison to the logical circuit described in [36] with all the CNOT operations replaced with the best optical gates.

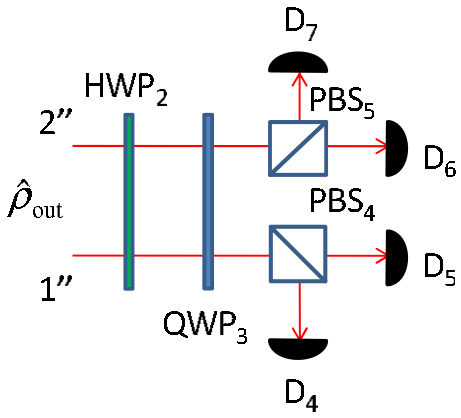

Figure 2. The final part of the setup that can be used for verification of the cloning operation. The HWP_2 and QWP_3 rotate $|\psi\rangle$ and $|\overline{\psi}\rangle$ to $|1_{\text{H}}\rangle$ and $|1_{\text{V}}\rangle$. Cloning is successful when two of the detectors click, one of the pair (D_4, D_5) and one of the pair (D_6, D_7). Four detectors are used in order to evaluate the fidelity of the cloning operations as given in (18), where the pairs of detectors (D_4, D_5) and (D_6, D_7) correspond to the POVMs ($\hat{\overline{\Pi}}_1$, $\hat{\Pi}_1$) and ($\hat{\Pi}_1$,$\hat{\overline{\Pi}}_1$), respectively.

The setup proposed in this paper is suitable not only for the MPCC, but also for any optimal cloning of an arbitrary set of qubits of axial and mirror xy symmetry [37] including the universal, phase-covariant and mirror-phase covariant clonings.

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