Entanglement and nonclassicality of twin beams containing noise

Arkhipov I.,[a] Peřina J., Jr.,[b] Miranowicz A.[c] and Peřina J.[a]

[a] Joint Laboratory of Optics of Palacký University and Institute of Physics of AS CR, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic;
[b] Joint Laboratory of Optics of Palacký University and Institute of Physics of AS CR, Institute of Physics of Academy of Science of the Czech Republic, 17. listopadu 50a, 771 46 Olomouc, Czech Republic;
[c] Faculty of Physics, Adam Mickiewicz University, PL-61-614 Poznan, Poland

ABSTRACT

Using the characteristic function of a twin beam composed of both paired and noise components we derive the corresponding statistical operator in the Fock-state basis. Applying the Peres-Horodecki criterion for a partially transposed statistical operator, we determine the negativity of the twin beam to quantify entanglement. In parallel, nonclassicality of the twin beam is quantified by nonclassical depth which is the acceptable amount of noise photons that preserves non-classicality manifested by negative values of the Glauber-Sudarshan quasi-probability function. The connection between entanglement and non-classicality is discussed considering the noise present either in one or both fields constituting the twin beam. Also the state of dimensionality via the Schmidt number as well as von Neumann entropy of the twin beam is analyzed.

Keywords: twin beam, non-classicality, entanglement, dimensionality.

1. INTRODUCTION

Nonclassicality and entanglement are at the heart of quantum optics that studies optical fields with no classical counterparts. Nonclassical properties of such fields have been found useful both for elucidating the rules governing the nature and various applications, e.g., in the field of quantum information processing,[1] quantum metrology[2] or highly-sensitive measurements.[3] The answer to the question whether a given state can be described within a classical theory has been considered as one of the most serious problems since the early days of quantum physics.[4–6] From both theoretical and experimental point of view, the nonlinear process of optical parametric down conversion with its production of photon pairs has played an important role here.[7,8] It has allowed to generate both light with its quadratures squeezed below the vacuum level[9,10] as well as light with sub-Poissonian photon-number statistics.[11,12] On the other hand entanglement has been exploited in observing non-locality of quantum physics as well as ruling out any neoclassical physical theory using the Bell inequalities[13]

In quantum optics, the definition of nonclassicality is based upon the Glauber-Sudarshan P representation[14,15] of the statistical operator of a given field. The commonly accepted formal criterion for distinguishing nonclassical states from the classical ones reads as follows:[7,16–18] A quantum state is *nonclassical* if its Glauber-Sudarshan P function fails to have the properties of a probability density. Alternatively, several operational criteria for nonclassicality of either single-mode[16,17,19,20] or multi-mode fields are at disposal. Their derivations are based either on field's moments[20,21] or direct reconstruction of integrated-intensity probability quasi-distribution.[7]

Entanglement is a special nonclassical property that describes quantum correlations among in general several subsystems. Such correlations cannot be treated by the means of classical statistical theory and so they are considered as nonclassical. To detect entanglement several approaches have been developed for discrete and continuous variables. Entangled states belong to non-separable states. This property has been exploited in suggesting entanglement criteria based on the partial transposition of a statistical operator.[22–24] Another approach

E-mail: arkhipov@jointlab.upol.cz

19th Polish-Slovak-Czech Optical Conference on Wave and Quantum Aspects of Contemporary Optics, edited by Agnieszka Popiołek-Masajada, Wacław Urbańczyk, Proc. of SPIE Vol. 9441, 94410Y
 · doi: 10.1117/12.2086608

has been based on the violation of the Bell inequalities written for different mean values comprising the measurement on both parts of a bipartite system.[25] Also a method using positive semi-definite matrices containing moments of different orders[20,21] has been found very powerful. We note that entanglement is the very crucial tool in today's quantum information processing.

In this contribution, we determine nonclassicality as well as entanglement for twin beams with different intensities containing both photon pairs and single photons. Such fields are provided in general by parametric down-conversion that emits together a signal and an idler field. These fields thus naturally form a bipartite quantum system.

This paper is organized as follows. In Sec. II, the model of parametric down-conversion giving an appropriate statistical operator for twin beams is presented. Entanglement of twin beams is addressed in Sec. III using negativity. Criteria for nonclassicality are discussed in Sec. IV. The occurrence of both nonclassicality and entanglement for twin beams with different parameters is studied in Sec. V. Effective dimensionality of statistical operators of twin beams quantified by the Schmidt number is determined in Sec. VI, together with entropy. Conclusions are drawn in Sec. VII.

2. QUANTUM MODEL OF A TWIN BEAM

In order to describe the generation of a twin beam by two-mode parametric down-conversion, we use the approach based on the Heisenberg equations with an appropriate Hamiltonian:[7]

$$\hat{H}_{\rm int} = -\hbar g \hat{A}_1 \hat{A}_2 \exp({\rm i}\omega t - {\rm i}\phi) + {\rm H.c.} \tag{1}$$

Symbols $\hat{A}_1$ ($\hat{A}_1^\dagger$) and $\hat{A}_2$ ($\hat{A}_2^\dagger$) represent annihilation (creation) operators of the signal and idler beam, respectively, g is a real coupling constant being linearly proportional to the quadratic susceptibility and the real amplitude of pumping. Interaction time is denoted as t, ω (ϕ) is the pump-beam frequency (phase), ω_1 and ω_2 stand for the signal and idler frequencies, respectively. The energy conservation law provides the relation $\omega = \omega_1 + \omega_2$. Symbol H.c. means the Hermitian conjugated term. In the real nonlinear process, also noise occurs. It can be described by the Langevin forces $\hat{L}$ belonging to a reservoir of chaotic oscillators with the mean number of noise photons $\langle n_d \rangle$.

The Heisenberg-Langevin equations corresponding to Hamiltonian $\hat{H}_{\rm int}$ are written as

$$\begin{aligned} \frac{{\rm d}\hat{A}_1}{{\rm d}t} &= -{\rm i}\omega_1 \hat{A}_1 + {\rm i}g\hat{A}_2^\dagger \exp(-{\rm i}\omega t + {\rm i}\phi) + \hat{L}_1, \\ \frac{{\rm d}\hat{A}_2}{{\rm d}t} &= -{\rm i}\omega_2 \hat{A}_2 + {\rm i}g\hat{A}_1^\dagger \exp(-{\rm i}\omega t + {\rm i}\phi) + \hat{L}_2. \end{aligned} \tag{2}$$

The Langevin operators $\hat{L}_i$, $i = 1, 2$, have the following properties:

$$\langle \hat{L}_i \rangle = \langle \hat{L}_i^\dagger \rangle = 0, \qquad \langle \hat{L}_i^\dagger \hat{L}_j \rangle = \delta_{ij} \langle n_d \rangle, \qquad \langle \hat{L}_i \hat{L}_j^\dagger \rangle = \delta_{ij} \left(\langle n_d \rangle + 1 \right), \tag{3}$$

where δ_{ij} stands for the Kronecker symbol.

Using the interaction representation $[\hat{A}_j(t) = a_j(t)\exp(-i\omega_j t)]$ and neglecting the Langevin forces, the solution of (2) attains the form

$$\begin{aligned} \hat{a}_1(t) &= \hat{a}_1(0)u(t) + i\hat{a}_2^\dagger(0)v(t)\exp(i\phi), \\ \hat{a}_2(t) &= \hat{a}_2(0)u(t) + i\hat{a}_1^\dagger(0)v(t)\exp(i\phi), \end{aligned} \tag{4}$$

where $u(t) = \cosh(gt)$ and $v(t) = \sinh(gt)$.

In the next step, we calculate the normal characteristic function $C_{\mathcal{N}}$:

$$C_{\mathcal{N}}(\beta_1, \beta_2) = {\rm Tr}\left[\hat{\rho}\exp(\beta_1 \hat{a}_1^\dagger + \beta_2 \hat{a}_2^\dagger)\exp(-\beta_1^* \hat{a}_1 - \beta_2^* \hat{a}_2)\right]; \tag{5}$$

where Tr means the trace. Using the solution (4), the normal characteristic function $C_{\mathcal{N}}$ attains the Gaussian form:[26]

$$C_{\mathcal{N}}(\beta_1, \beta_2) = \exp\left[-(|\beta_1|^2 B_1 + |\beta_2|^2 B_2) + D_{12}\beta_1^*\beta_2^* + D_{12}^*\beta_1\beta_2\right] \tag{6}$$

in which β_1 and β_2 denote variables of this function. For the noiseless case, we have $B_1 = \langle \triangle\hat{a}_1^\dagger \triangle \hat{a}_1\rangle = B_2 = \langle \triangle\hat{a}_2^\dagger \triangle \hat{a}_2\rangle = B = |v|^2 = \sinh^2(gt)$ and $D_{12} = \langle \triangle\hat{a}_1 \triangle \hat{a}_2\rangle = uv = \mathrm{i}/2\sinh(2gt)\exp(\mathrm{i}\phi)$. When noise is also considered, the parameter B contains an additional noise contribution B_N, i.e. $B' = B + B_N$. Whereas the parameter B gives the mean number of photon pairs, the parameter B_N equals the mean number of noise photons coming from the reservoir ($B_N = \langle n_d\rangle$). On the other hand, the parameter D_{12} is not influenced by the noise.

The statistical operator $\hat{\rho}$ of a twin beam then acquires the following form:[7]

$$\hat{\rho} = \frac{1}{\pi^2}\int d^2\beta_1 d^2\beta_2 \mathcal{C}_{\mathcal{A}}(\beta_1, \beta_2) : \exp\left(\sum_{j=1}^{2} \hat{a}_j\beta_j^* - \hat{a}_j^\dagger\beta_j\right) :; \tag{7}$$

where $C_{\mathcal{A}}(\beta_1, \beta_2) = C_{\mathcal{N}}(\beta_1, \beta_2)\exp(-|\beta_1|^2 - |\beta_2|^2)$ denotes an anti-normal characteristic function and symbol :: means normal ordering of field operators.

For the case with equal amount of noise in twin beams, the integral in (7) takes the form

$$\hat{\rho} = \frac{1}{K} : \exp\left[-\frac{B'}{K}\hat{a}_1^\dagger\hat{a}_1 - \frac{B'}{K}\hat{a}_2^\dagger\hat{a}_2 + \frac{|D_{12}|}{K}\left(\hat{a}_1\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_2^\dagger\right)\right] : \tag{8}$$

where $K = B'^2 - |D_{12}|^2$. Decomposing the obtained statistical operator $\hat{\rho}$ in the Fock basis we finally arrive at the formula

$$\langle ij|\hat{\rho}|kl\rangle = \sum_{n,p,r,t=0}^{\infty} \frac{1}{n!}(-1)^{n-r}\binom{n}{p}\binom{p}{r}\binom{r}{t} B'^{n-r} K^{-n-1} |D_{12}|^r \langle ij|\hat{a}_1^{\dagger n-p+t}\hat{a}_2^{\dagger p-r+t}\hat{a}_1^{n-p+r-t}\hat{a}_2^{p-t}|kl\rangle. \tag{9}$$

Similar formulas can be obtained for the case with noise present only in one beam.

3. NEGATIVITY OF A TWIN BEAM

Negativity $\mathcal{N}$ of a mixed bipartite system defined on the basis of the Peres-Horodecki criterion[22–24] is useful for quantifying entanglement of a twin beam. It can be expressed as

$$\mathcal{N}(\hat{\rho}) = \frac{||\hat{\rho}^{PT}||_1 - 1}{2} \tag{10}$$

using a trace norm $||\rho^{PT}||_1$ of the partially transposed statistical operator ρ^{PT}. Negativity essentially measures the degree at which ρ^{PT} fails to be positive. Therefore it can be regarded as a quantitative version of the Peres-Horodecki criterion for separability.[22,23] Following (10), the negativity $\mathcal{N}$ is given as the absolute value of the sum of negative eigenvalues of ρ^{PT}. It vanishes for un-entangled states. It is worth noting that negativity $\mathcal{N}(\rho)$ is an entanglement monotone and so it can be used to quantify the degree of entanglement in bipartite systems. We have analyzed the dependence of negativity $\mathcal{N}(\rho)$ on mean photon-pair and noise photon numbers B and B_N in two specific physically interesting situations. In the first case, the same amount B_N of noise photons was present in both the signal and idler beams (see Fig. 1). On the other hand, noise photons occurred only in one (e.g., the signal) beam whereas the other beam contained only paired photons (see Fig. 2) in the second analyzed case. In both cases, twin beams with varying mean photon-pair numbers B and different (but fixed) ratios of the mean noise photon numbers B_N and mean photon-pair numbers B (noise-to-signal ratio) have been investigated.

As follows from the graphs in Figs.1 and 2, negativity $\mathcal{N}(\hat{\rho})$ increases with the increasing mean photon-pair number B for the noiseless case as well as for smaller values of B such that $B_N < 0.5$ for the first case and $B_N < 1$ for the second case. It holds that the larger the ratio B_N/B, the smaller the values of negativity $\mathcal{N}$. As follows from the comparison of curves in Figs.1 and 2, the presence of noise in both beams results in lower values of negativity $\mathcal{N}$. In general, negativity $\mathcal{N}(\hat{\rho})$ shows strong sensitivity to noise. Even a small amount of noise can spoil the entanglement regardless the value of mean photon-pair number B.

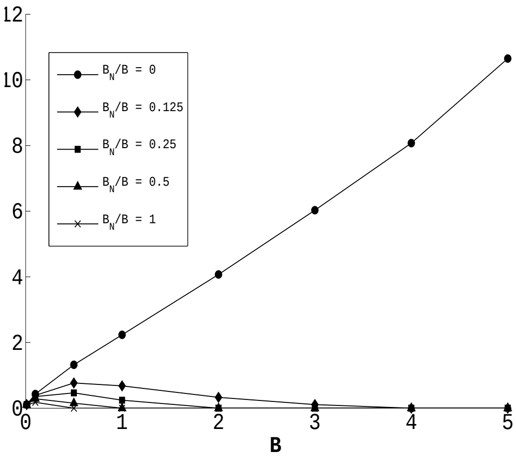

Figure 1. Negativity $\mathcal{N}$ as a function of mean photon-pair number B for different ratios B_N/B assuming noise B_N in both signal and idler beams.

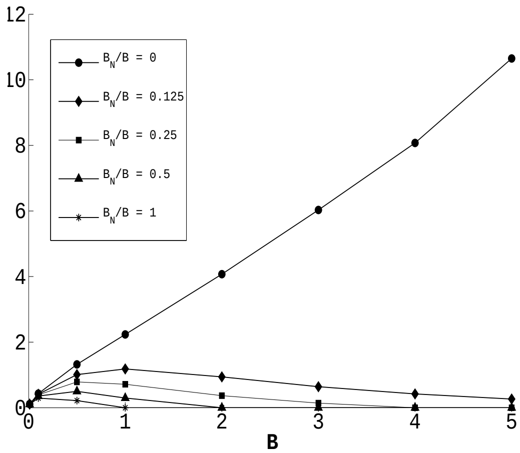

Figure 2. Negativity $\mathcal{N}$ as a function of mean photon-pair number B for different ratios B_N/B assuming noise B_N in the signal beam only.

4. NONCLASSICALITY OF A TWIN BEAM

We quantify nonclassicality according to nonclassicality depth r[27] which is related to the threshold value[26, 28] $s_{\rm th}$ of the ordering parameter s via expression $r = (1 - s_{\rm th})/2$. It holds that the joint signal-idler quasi-distribution of integrated intensities attains negative values for $r > 0$ ($1 \geq s > s_{\rm th}$) and thus describes a nonclassical field. Considering the case with noise in both beams, nonclassicality depth r is determined along the formula[28]

$$r = \sqrt{B + B^2} - B - B_N. \tag{11}$$

On the other hand, formula

$$r = \sqrt{B + B^2 + \frac{B_N^2}{4}} - B - \frac{B_N}{2} \tag{12}$$

is appropriate for the case with noise only in the signal beam. As formula (11) suggests, nonclassicality depth r gives the mean number of noise photons which adding into the nonclassical field removes its nonclassicality. So, the larger the value of nonclassicality depth r is, the more nonclassical the field is. On the other hand, formal application of formula (11) to classical noisy twin beams results in negative values of nonclassicality depth r. Their absolute value $|r|$ can be considered as a measure of classicality of noisy twin beams in the sense that it quantifies the mean number of photon pairs needed for transforming a classical twin beam to the classical-quantum border $r = 0$ ($\sqrt{B + B^2} - B$).

5. COMPARISON OF ENTANGLEMENT AND NONCLASSICALITY

Our results have shown that entanglement and nonclassicality closely accompany each other in a twin beam. In a noiseless twin beam, the more intense the twin beam is, the more nonclassical the twin beam is and also the more entangled the signal and idler beams are. As follows from formulas (11) and (12) for a fixed ratio B_N/B, nonclassicality depth r increases with the increasing mean photon-pair number B first and, after reaching certain value of B, it decreases and finally reaches zero values characterizing a classical twin beam. It can be shown that for $B_N \geq 1/2$ ($B_N \geq 1$) for the noise in both beams (only one beam), the twin beam is classical. This behavior resembles that of negativity $\mathcal{N}$ documented by the curves in Figs. 1 and 2. This similarity in the behaviors of nonclassicality depth r and negativity $\mathcal{N}$ is very close, as revealed by the curves showing the dependence of negativity $\mathcal{N}$ on nonclassicality depth r in Figs. 3 and 4. The curves obtained for different values of noise-to-signal ratio B_N/B lie on the curve characterizing the noise twin beams. This means that there exists a monotone mapping between nonclassicality depth r and negativity $\mathcal{N}$ for the analyzed twin beams. From the point of view of noise in a twin beam, it quantitatively equally degrades both nonclassicality depth r and negativity $\mathcal{N}$. We can say in general, that pairing of photons creates entanglement between the signal and idler beams and the presence of entanglement in a twin beam is then responsible for its nonclassical behavior observed for the overall twin beam (without considering its partitioning into the signal and idler beams).

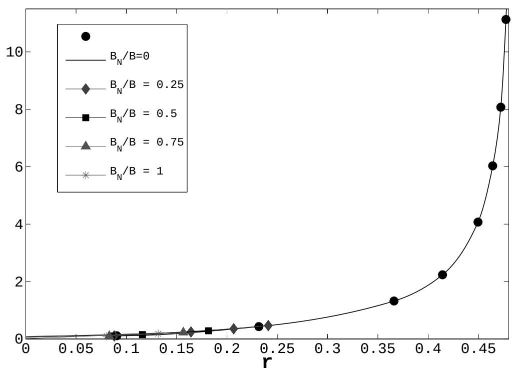

Figure 3. Negativity $\mathcal{N}$ in dependence on nonclassicality depth r for twin beams plotted in Fig. 1.

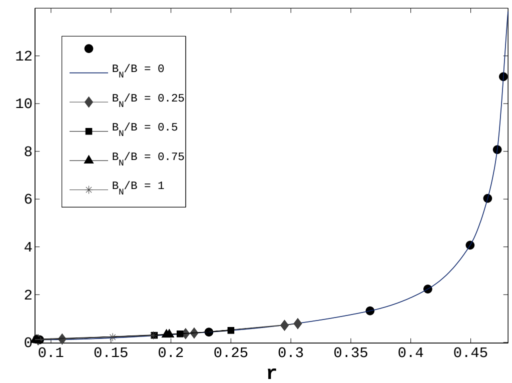

Figure 4. Negativity $\mathcal{N}$ in dependence on nonclassicality depth r for twin beams plotted in Fig. 2.

6. DIMENSIONALITY AND ENTROPY OF A TWIN BEAM

Dimensionality of a bipartite entangled state is given by the Schmidt number determined from eigenvalues λ_j of the reduced statistical operators describing the signal and idler beams solely. The appropriate formula is written as

$$K = \frac{(\mathrm{Tr}_1[\hat{\rho}_1])^2}{\mathrm{Tr}_1[\hat{\rho}_1^2]} = \frac{(\mathrm{Tr}_2[\hat{\rho}_2])^2}{\mathrm{Tr}_2[\hat{\rho}_2^2]} = \frac{\left(\sum_j \lambda_j\right)^2}{\sum_j \lambda_j^2}; \tag{13}$$

where $\hat{\rho}_1 = \mathrm{Tr}_2[\hat{\rho}]$ and $\hat{\rho}_2 = \mathrm{Tr}_1[\hat{\rho}]$. The larger the Schmidt number K, the greater the number of paired eigenmodes needed for building the twin beam and the more complex the correlations between the signal and idler beams.

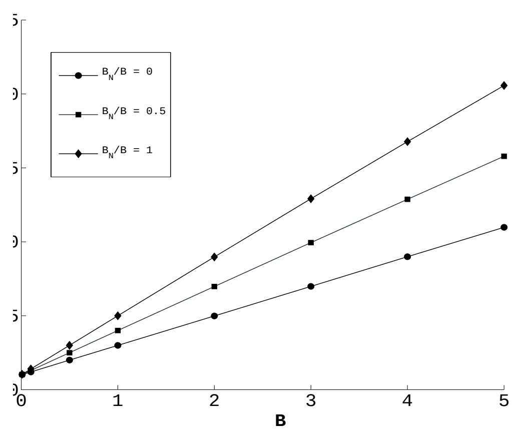

Figure 5. The Schmidt number K as a function of mean photon-pair number B for different ratios B_N/B assuming noise B_N in both signal and idler beams.

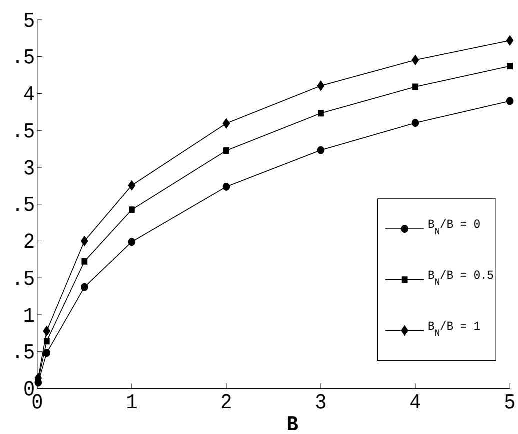

Figure 6. The von Neumann entropy S in dependence on mean photon-pair number B for different ratios B_N/B assuming noise B_N in both signal and idler beams.

Bipartite correlations in a twin beam can also be quantified by the von Neumann entropy S of the reduced statistical operators determined by the formula:

$$S = -\sum_j \lambda_j \log \lambda_j. \tag{14}$$

Note that for pure states, the entropy S is a good measure of entanglement.

Considering twin beams with noise in both beams, both the Schmidt number K (see Fig. 5) as well as entropy S (see Fig. 6) increase with the increasing mean photon-pair number B and the mean noise photon number B_N. Roughly speaking, the more intense is the twin beam, the greater are the Schmidt number K and entropy S.

The comparison of curves for negativity $\mathcal{N}$ in Fig. 1 with those for the Schmidt number K in Fig. 5 and entropy S in Fig. 6 reveals that neither the Schmidt number K nor entropy S are suitable for the quantification of entanglement of mixed states in general for the noisy twin beams. Additional noise in the signal and idler beams results in certain 'unbalance' of the original maximally entangled state that requires more dimensions in its description. However, the noise naturally decreases entanglement between the signal and idler beams.

7. CONCLUSIONS

We have quantified nonclassicality (via nonclassicality depth) as well as entanglement (using negativity) of both noiseless and noisy twin beams. It has been shown that both entanglement and nonclassicality are sensitive to noise. This noise cannot exceed certain value for their observation. We have also demonstrated that entanglement and nonclassicality of twin beams are closely related. Nonclassicality of a twin beam occurs as a consequence of entanglement between its signal and idler parts originating in pairing of photons in the twin beam. For comparison, the Schmidt number as a measure of dimensionality and the von Neumann entropy of twin beams have been determined in parallel with negativity. Whereas additional noisy photons increase the Schmidt number and entropy, they decrease negativity.

ACKNOWLEDGMENTS

Support by projects CZ.1.05/2.1.00/03.0058 and CZ.1.07/2.3.00/20.0017 of MŠMT ČR is acknowledged. I.A. thanks project IGA_PrF_2014005 of IGA UP Olomouc. J.P. acknowledges project P205/12/0382 of GA ČR.

REFERENCES

1. Nielsen M. A. and Chuang I. L., *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, 2000.
2. Giovannetti V., Lloyd S. and Maccone L., "Advances in quantum metrology," *Nat. Photon.* **5**, p. 222, 2011.
3. Vahlbruch H., Khailaidovski A., Lastzka N., Graef C., Danzmann K. and Schnabel R., "Squeezed light at 1550nm with a quantum noise reduction of 12.3db," *Class. Quantum Grav.* **27**, p. 084027, 2010.
4. Einstein A., "Über einen die erzeugung und verwandlung des lichtes betreffenden heuristischen gesichtspunkt," *Ann. Phys.* **17**, p. 132, 1905.
5. Schrodinger E., "Die gegenwartige situation in der. quantenmechanik," *Naturwissenschaften* **23**, p. 807, 1935.
6. Einstein A., Podolsky B. and Rosen N., "Can quantum-mechanical description of physical reality be considered complete?," *Phys. Rev.* **47**, p. 777, 1935.
7. Peřina J., *Quantum Statistics of Linear and Nonlinear Optical Penonomena*, Kluwer Academic Publishers, Dodrecht, The Netherlands, second completely revised ed., 1991.
8. Walls D. F. and Milburn G. J., *Quantum Optics*, Springer, Berlin, 1994.
9. A. Lukš and V. Peřinová and J. Peřina, "Principal squeezing of vacuum fluctuations," *Opt. Commun.* **67**, pp. 149—151, 1988.
10. A. I. Lvovsky and M. G. Raymer, "Continuous-variable optical quantum state tomography," *Rev. Mod. Phys.* **81**, pp. 299—332, 2009.
11. Bondani M., Alevi A., Zambra G., Paris M. G. A., and Andreoni A., "Sub-shot-noise photon-number correlation in a mesoscopic twin beam of light," *Phys. Rev. A* **76**, p. 013833, 2007.

12. Peřina J., Jr., Haderka O., Michálek V., "Sub-poissonian-light generation by postselection from twin beams," *Optics Express* **21**(16), p. 19387, 2013.
13. G. Weihs and T. Jennewein and C. Simon and H. Weinfurter and A. Zeilinger, "Violation of bell's inequality under strict einstein locality conditions," *Phys. Rev. Lett.* **81**(23), pp. 5039—5043, 1998.
14. Glauber R.J., "Coherent and incoherent states of the radiation field," *Phys. Rev.* **131**, p. 2766, 1963.
15. Sudarshan E.C.G., "Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams," *Phys. Rev. Lett.* **10**, p. 277, 1963.
16. Dodonov V. and Manko V., *Theory of Nonclassical States of Light*, Taylor & Francis, New York, 2003.
17. Vogel W. and Welsch D., *Quantum Optics*, Wiley-VCH, Weinheim, 2006.
18. Mandel L. and Wolf E., *Optical Coherence and Quantum Optics*, Cambridge University Press, Cambridge, 1995.
19. Richter Th. and Vogel W., "Nonclassicality of quantum states: A hierarchy of observable conditions," *Phys. Rev. Lett.* **89**, p. 283601, 2002.
20. Miranowicz A., Bartkowiak M., Wang X., Liu X. and Nori F., "Testing nonclassicality in multimode fields: A unified derivation of classical inequalities," *Phys. Rev. A* **82**, p. 013824, 2010.
21. Vogel W., "Nonclassical correlation properties of radiation fields," *Phys. Rev. Lett.* **100**, p. 013605, 2008.
22. Peres A., "Separability criterion for density matrices," *Phys. Rev. Lett.* **77**, p. 1413, 1996.
23. Horodecki P., "Separability criterion and inseparable mixed states with positive partial transposition," *Phys. Lett. A* **232**, p. 333, 1997.
24. Vidal G. and Werner R.F., "Computable measure of entanglement," *Phys. Rev. A* **65**, p. 032314, 2002.
25. Clauser J. F., Horne M. A., Shimony A. and Holt R. A., "Proposed experiment to test local hidden-variable theories," *Phys. Rev. Lett.* **23**(15), pp. 880–884, 1969.
26. Peřina J. and Křepelka J., "Multimode description of spontaneous parametric down-conversion," *J. Opt. B: Quantum Semiclass. Opt.* **7**, pp. 246–252, 2005.
27. Lee C.T., "Measure of the nonclassicality of nonclassical states," *Phys. Rev. A* **44**, p. R2775, 1991.
28. Peřina J., Jr., Haderka O., Michálek V. and Hamar M., "State reconstruction of a multimode twin beam using photodetection," *Phys. Rev. A* **87**, p. 022108, 2013.