Quantifying Non-Markovianity with Temporal Steering: Supplementary Material

Shin-Liang Chen,[1] Neill Lambert,[2] Che-Ming Li,[3] Adam Miranowicz,[2,4] Yueh-Nan Chen,[1,2,*] and Franco Nori[2,5]

[1]*Department of Physics and National Center for Theoretical Sciences, National Cheng-Kung University, Tainan 701, Taiwan*
[2]*CEMS, RIKEN, 351-0198 Wako-shi, Japan*
[3]*Department of Engineering Science and Supercomputing Research Center, National Cheng-Kung University, Tainan City 701, Taiwan*
[4]*Faculty of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland*
[5]*Department of Physics, The University of Michigan, Ann Arbor, MI 48109-1040, USA*

In this supplementary material we give a few illustrative examples of the calculation of the temporal steerable weight and its application as a measure of strong non-Markovianity for some prototype models.

HOW TO CALCULATE THE STEERABLE WEIGHT: A PEDAGOGICAL EXAMPLE

Here we show explicitly how to calculate the steerable weight of Skrzypczyk *et al.* [1] in a simple example. Specifically, we assume three types of measurements corresponding to the projections on the eigenstates of the Pauli operators:

$$\begin{aligned} X &= |+\rangle\langle+|-|-\rangle\langle-|, \\ Y &= |R\rangle\langle R|-|L\rangle\langle L|, \\ Z &= |0\rangle\langle 0|-|1\rangle\langle 1|, \end{aligned} \tag{1}$$

where $|0\rangle = |H\rangle$, $|1\rangle = |V\rangle$, $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, $|R\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$, and $|L\rangle = (|0\rangle - i|1\rangle)/\sqrt{2}$, which can be interpreted as: horizontal, vertical, diagonal, antidiagonal, right-circular, and left-circular polarization states for the optical polarization qubits, respectively. We can label the eigenstates of the Pauli operators together with their eigenvalues as follows: $|x_1\rangle = |+\rangle$ with $x_1 = +1$, $|x_2\rangle = |-\rangle$ with $x_2 = -1$, $|y_1\rangle = |+\rangle$ with $y_1 = +1, \ldots$, and $|z_2\rangle = |1\rangle$ with $z_2 = -1$.

Then, possible unnormalized states of Bob $\sigma_{a|x}$ $(x = X, Y, Z)$ for a given two-qubit state ρ read

$$\begin{aligned} \sigma^{(1)}_{a|x} &\equiv \sigma_{+1|X} = \mathrm{Tr}_A[(|+\rangle\langle+|\otimes I)\rho], \\ \sigma^{(2)}_{a|x} &\equiv \sigma_{-1|X} = \mathrm{Tr}_A[(|-\rangle\langle-|\otimes I)\rho], \\ \sigma^{(3)}_{a|x} &\equiv \sigma_{+1|Y} = \mathrm{Tr}_A[(|R\rangle\langle R|\otimes I)\rho], \\ \sigma^{(4)}_{a|x} &\equiv \sigma_{-1|Y} = \mathrm{Tr}_A[(|L\rangle\langle L|\otimes I)\rho], \\ \sigma^{(5)}_{a|x} &\equiv \sigma_{+1|Z} = \mathrm{Tr}_A[(|0\rangle\langle 0|\otimes I)\rho], \\ \sigma^{(6)}_{a|x} &\equiv \sigma_{-1|Z} = \mathrm{Tr}_A[(|1\rangle\langle 1|\otimes I)\rho], \end{aligned} \tag{2}$$

where I is the single-qubit identity operator. A classical random variable held by Alice,

$$\lambda_n = [x_i, y_j, z_k] \equiv [\langle x_i|X|x_i\rangle, \langle y_i|Y|y_i\rangle, \langle z_i|Z|z_i\rangle], \tag{3}$$

can take the following values:

$$\begin{aligned} \lambda_1 &= [-1,-1,-1], \quad \lambda_2 = [-1,-1,+1], \\ \lambda_3 &= [-1,+1,-1], \quad \lambda_4 = [-1,+1,+1], \\ \lambda_5 &= [+1,-1,-1], \quad \lambda_6 = [+1,-1,+1], \\ \lambda_7 &= [+1,+1,-1], \quad \lambda_8 = [+1,+1,+1]. \end{aligned} \tag{4}$$

The extremal deterministic single-party conditional probability distributions for Alice read

$$\begin{aligned} [D_{\lambda_1}(+1|X), \ldots, D_{\lambda_8}(+1|X)] &= [0,0,0,0,1,1,1,1], \\ [D_{\lambda_1}(-1|X), \ldots, D_{\lambda_8}(-1|X)] &= [1,1,1,1,0,0,0,0], \\ &\vdots \\ [D_{\lambda_1}(-1|Z), \ldots, D_{\lambda_8}(-1|Z)] &= [1,0,1,0,1,0,1,0]. \end{aligned} \tag{5}$$

Let us denote an unsteerable assemblage as

$$\sigma^{\mathrm{US}}_{a|x} \equiv \sum_\lambda D_\lambda(a|x)\sigma_\lambda = \sum_{n=1}^{8} D_{\lambda_n}(a|x)\sigma_{\lambda_n}. \tag{6}$$

Then, we have

$$\begin{aligned} \sigma^{(1)\mathrm{US}}_{a|x} &\equiv \sigma^{\mathrm{US}}_{+1|X} = \sigma_{\lambda_5} + \sigma_{\lambda_6} + \sigma_{\lambda_7} + \sigma_{\lambda_8}, \\ \sigma^{(2)\mathrm{US}}_{a|x} &\equiv \sigma^{\mathrm{US}}_{-1|X} = \sigma_{\lambda_1} + \sigma_{\lambda_2} + \sigma_{\lambda_3} + \sigma_{\lambda_4}, \\ \sigma^{(3)\mathrm{US}}_{a|x} &\equiv \sigma^{\mathrm{US}}_{+1|Y} = \sigma_{\lambda_3} + \sigma_{\lambda_4} + \sigma_{\lambda_7} + \sigma_{\lambda_8}, \\ \sigma^{(4)\mathrm{US}}_{a|x} &\equiv \sigma^{\mathrm{US}}_{-1|Y} = \sigma_{\lambda_1} + \sigma_{\lambda_2} + \sigma_{\lambda_5} + \sigma_{\lambda_6}, \\ \sigma^{(5)\mathrm{US}}_{a|x} &\equiv \sigma^{\mathrm{US}}_{+1|Z} = \sigma_{\lambda_2} + \sigma_{\lambda_4} + \sigma_{\lambda_6} + \sigma_{\lambda_8}, \\ \sigma^{(6)\mathrm{US}}_{a|x} &\equiv \sigma^{\mathrm{US}}_{-1|Z} = \sigma_{\lambda_1} + \sigma_{\lambda_3} + \sigma_{\lambda_5} + \sigma_{\lambda_7}. \end{aligned} \tag{7}$$

The steerable weight SW can be given as the solution of the following semidefinite program: Find

$$\mathrm{SW} = 1 - \max \mathrm{Tr}\Big(\sum_{n=1}^{8} \sigma_{\lambda_n}\Big) \tag{8}$$

such that

$$\Big(\sigma^{(i)}_{a|x} - \sigma^{(i)\mathrm{US}}_{a|x}\Big) \geq 0 \quad \text{and} \quad \sigma_{\lambda_n} \geq 0 \tag{9}$$

for $i = 1, 2, \ldots, 6$ and $n = 1, \ldots, 8$. By using a numerical package for convex optimization [2–4], one can implement this semidefinite program in a straightforward way. This is easily generalized to the temporal case by replacing the two-qubit measurements in Eq. (2) with measurements on a single qubit, followed by evolution under the channel Λ.

EXAMPLE 1: COHERENT RABI OSCILLATIONS OF A MARKOVIAN SYSTEM

As a first simple example of the behavior of the temporal-SW under a Markovian dynamics, we consider a qubit that

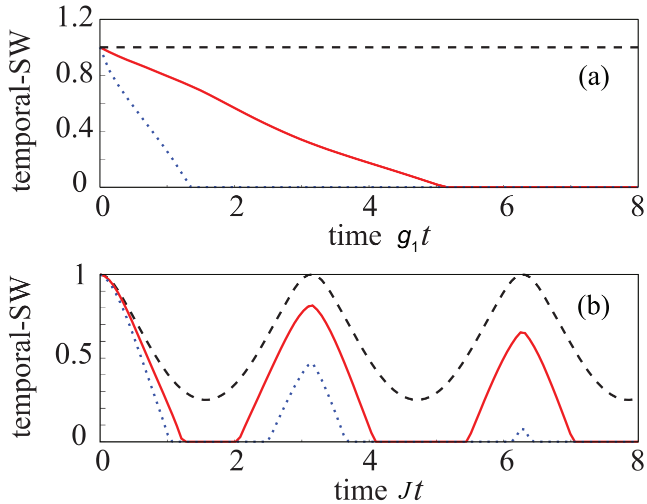

FIG. 1. (Color online) The temporal steerable weight (temporal-SW) as a function of evolution time when a system is in (a) a Markovian environment (example 1) and (b) non-Markovian environment (example 2). (a) The temporal-SW when the system undergoes coherent Rabi oscillations and purely Markovian decay (example 1). The black dashed, red solid, and blue dotted curves represent the results of the decay rate $\gamma_1/g_1 = 0$, $1/6$, and 1, respectively. The time t is in units of $1/g_1$, and $\hbar$ is set to 1. (b) The temporal-SW when the system interacts with a non-Markovian environment (example 2). The black dashed, red solid, and blue dotted curves represent the results of the decay rate $\gamma_2/J = 0$, 0.03, and 0.1, respectively. Here, the time t is in units of $1/J$.

undergoes coherent Rabi oscillations and purely Markovian decay. The Hamiltonian of the system is

$$H = \hbar g_1(\sigma_+ + \sigma_-), \tag{10}$$

where $\hbar g_1$ is the coherent coupling strength between two eigenstates, $|+\rangle$ and $|-\rangle$, of the qubit, and $\sigma_+ = |+\rangle\langle-|$ and $\sigma_- = |-\rangle\langle+|$ can be considered the raising and lowering operators, respectively. A Markovian channel induces a dissipation rate γ_1 from $|+\rangle$ to $|-\rangle$. We assume that the initial state, ρ_0 in Fig. 1 of the main text, is a maximally-mixed state and then perform projective measurements $M_{a|x}$ in three (or two) mutually-unbiased bases: $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ (or $\hat{X}$ and $\hat{Z}$). In Fig. 1(a), we plot the temporal-SW as a function of the evolution time t. We can see that the temporal-SW always remains the maximal value of unity if there is no decay, while the temporal-SW decreases monotonically when γ_1 is non-zero, as expected; the dynamics of this system is Markovian.

EXAMPLE 2: A SIMPLE NON-MARKOVIAN MODEL: A QUBIT COHERENTLY COUPLED TO ANOTHER QUBIT

Our second example is that of a qubit coherently-coupled to another qubit. If we treat one qubit as the system and the other one as the environment (by tracing it out), we have a very simple example of a non-Markovian environment. The total Hamiltonian of the system in the interaction picture is

$$H_{\text{int}} = \hbar J(\sigma_+^1\sigma_-^2 + \sigma_-^1\sigma_+^2), \tag{11}$$

where σ_+^i and σ_-^i are the raising and lowering operators of the ith qubit, and $\hbar J$ is the coherent coupling between the system and the environment. We assume the system qubit is also subject to an intrinsic decay with decay rate γ_2. In Fig. 1(b), we plot the temporal-SW for various decay rates γ_2, after tracing out the effective environment-qubit. The initial condition of the system-qubit is that of a maximally-mixed state, while the environment-qubit is in its excited state. As seen in Fig. 1(b), there is a vanishing and a reappearance of the temporal-SW of the system qubit. Since we know that the temporal-SW should decrease monotonically under a Markovian dynamics, the oscillation of temporal-SW naturally shows that the qubit is undergoing non-Markovian evolution. This memory effect in this simple example is easy to understand in that information regarding the state of the system-qubit flows to the environment-qubit and returns at a later time; one cannot assume that the evolution of the environment is not influenced by its history.

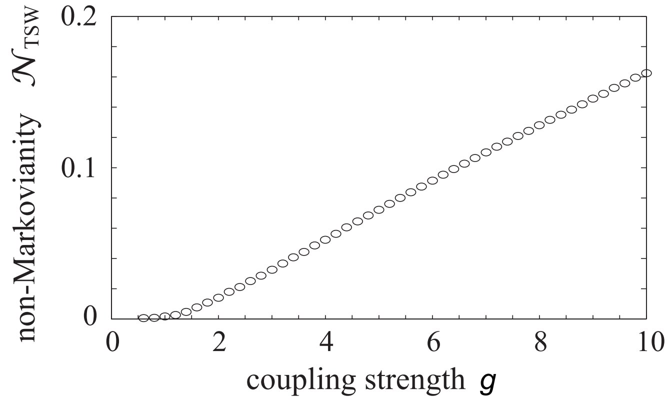

FIG. 2. (Color online) The degree of the non-Markovianity for a multimode reservoir with Lorentzian spectral density (example 3). The non-Markovianity $\mathcal{N}_{\text{TSW}}$, defined by the temporal steerable weight, as a function of the coupling strength g. Here, g is in units of spectral width ω_w.

EXAMPLE 3: A QUBIT COUPLED TO A NON-MARKOVIAN MULTIMODE RESERVOIR

In general, the dissipation γ rate in a Master equation description of an open-quantum system can be time-dependent, i.e. $\gamma = \gamma(t)$. If $\gamma(t) < 0$, it indicates that information can flow back to the system and the system dynamics can be non-Markovian. To show that the temporal-SW is sensitive to this, we use the same example as in Breuer *et al.* [5], where a qubit is coupled to a reservoir with a Lorentzian spectral density. In this case, the decay rate can be written as

$$\gamma(t) = -\frac{2}{G(t)}\frac{d}{dt}|G(t)|, \tag{12}$$

where

$$G(t) = e^{-\omega_w t/2}\left[\cosh\left(\frac{bt}{2}\right) + \frac{\omega_w}{b}\sinh\left(\frac{bt}{2}\right)\right] \tag{13}$$

with $b = \sqrt{\omega_w^2 - 2g\omega_w}$. Here, g denotes the coupling strength and ω_w is the spectral width. We choose a mixed state as the initial state and plot the non-Markovianity $\mathcal{N}_{\mathrm{TSW}}$ as a function of g/ω_w in Fig. 2. Our results agree well with those in Ref. [6]: the non-Markovianity is zero when $g/\omega_w < 0.5$, and increases monotonically as a function of g/ω_w.

* yuehnan@mail.ncku.edu.tw

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