





Liouvillian exceptional points of any order in dissipative linear bosonic systems: Coherence functions and switching between \mathcal{PT} and anti- \mathcal{PT} symmetries

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Usually, when investigating exceptional points (EPs) of an open Markovian bosonic system, one deals with spectral degeneracies of a non-Hermitian Hamiltonian (NHH), which can correctly describe the system dynamics only in the semiclassical regime. A recently proposed quantum Liouvillian framework [Minganti *et al.*, *Phys. Rev. A* **100**, 062131 (2019)] enables the complete determination of the dynamical properties of such systems and their EPs (referred to as Liouvillian EPs, or LEPs) in the quantum regime by taking into account the effects of quantum jumps, which are ignored in the NHH formalism. Moreover, the symmetry and eigenfrequency spectrum of the NHH become a part of much larger Liouvillian eigenspace. As such, the EPs of an NHH form a subspace of the LEPs. Here we show that once an NHH of a dissipative linear bosonic system exhibits an EP of a certain finite order n , it immediately implies that the corresponding LEP can become of any higher order $m \geq n$ defined in the infinite Hilbert space. These higher-order LEPs can be identified by the coherence and spectral functions at the steady state. The coherence functions can offer a convenient tool to probe extreme system sensitivity to external perturbations in the vicinity of higher-order LEPs. As an example, we study a linear bosonic system of a bimodal cavity with incoherent mode coupling to reveal its higher-order LEPs; particularly, of second and third order via first- and second-order coherence functions, respectively. Accordingly, these LEPs can be additionally revealed by squared and cubic Lorentzian spectral lineshapes in the power and intensity-fluctuation spectra. Moreover, we demonstrate that these EPs can also be associated with spontaneous parity-time (\mathcal{PT}) and anti- \mathcal{PT} -symmetry breaking in the system studied. These symmetries can be switched in the output fields (the so-called supermodes) of an additional linear coupler with a properly chosen coupling strength. Thus, we show that the initial loss-loss dynamics for the supermodes can be equivalent to the balanced gain-loss evolution.

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I. INTRODUCTION

A. Exceptional points

Exceptional points (EPs), which are exotic spectral degeneracies of non-Hermitian open systems, have attracted much interest in the last decades [1–3]. EPs arise when both eigenvalues and eigenfunctions of a non-Hermitian Hamiltonian (NHH) coalesce. Equivalently, an NHH, at its EPs, attains a Jordan form, i.e., it fails to be diagonalized. The advantages of EPs for applications remain a very active topic of research [4–18], in particular concerning the metrological advantages of EP sensitivity to external perturbations [1].

The concept of EPs was first introduced in connection with the perturbation theory of linear operators [19]. In physics,

the notion of the EPs was further explored in connection with parity-time-symmetric (\mathcal{PT} -symmetric) quantum mechanics [20]. More recently, the concept of EPs has been investigated for general open quantum systems, where the interplay of incoherent drives and losses with coherent coupling can lead to the observation of an NHH spectral degeneracy [1,2,21,22].

The presence of EPs in a system can produce a plethora of nontrivial phenomena. To name a few: unidirectional invisibility [23,24], lasers with enhanced-mode selectivity [25,26], low-power nonreciprocal light transmission [27–29], new types of thresholdless phonon lasers [30,31], enhanced light-matter interactions [8,9,11], loss-induced lasing [32,33], and even exceptional photon blockade [34]. These exotic phenomena have been observed in different experimental platforms, based on electronics [35], optomechanics [30,36–38], acoustics [39,40], plasmonics [41], and metamaterials [42]. Moreover, the concept of EPs has been successfully exploited in describing dynamical quantum phase transitions and topological phases of condensed matter in open quantum systems

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[43–51], and its relation to nonclassicality in photonic systems [52,53].

B. Liouvillian exceptional points

Recently, an extension of the concept of EPs from an NHH formalism to that based on the quantum Liouvillian has been proposed [54]. Indeed, the inclusion of quantum jumps can have a profound effect on the system dynamics and its spectra [54–59]. Moreover, the eigenspectrum of an NHH becomes a part of much larger eigenspace of a Liouvillian, meaning that EPs of an NHH, which from hereon we denote as HEPs, become a part of Liouvillian EPs which are usually denoted as LEPs. As such, the quantum Liouvillian formalism appears to be a natural choice to study EPs of an open quantum system. Nonetheless, in some systems, e.g., in a double-quantum-dot circuit QED setup [60], the NHH can capture nontrivial system dynamics, which is “invisible” when using the corresponding Liouvillian.

At the same time, very recent works have already shown that switching to the Liouvillian framework allows us to reveal nontrivial phenomena, which cannot be observed within the NHH formalism. These include the existence of higher-order LEPs compared with lower-order HEPs [56,61], or the possibility to detect nontrivial EPs by using a hybrid Liouvillian formalism [57]. Moreover, \mathcal{PT} symmetry has been reformulated within a Liouvillian framework [55,59,62].

C. Results

In this work, we demonstrate that, for a dissipative linear bosonic system whose effective NHH exhibits a HEP of any finite order n determined by first-order field moments, the corresponding LEPs can become at least of order $m \geq 2k(n-1) + 1$, for $k = \frac{1}{2}, 1, 2, 3, \dots$, determined by higher-order field moments, accordingly. These m th-order LEPs can be identified by the $(m-1)$ st-order coherence and spectral response functions in the steady state of the system. Because the steady state of such systems is a thermal state, by exploiting the well-known moments theorem [63], the coherence functions of arbitrary order can be completely determined by the first-order coherence function, i.e., by its products. As such, by identifying a HEP with the first-order coherence function [56], one then can identify higher-order LEPs by means of the corresponding higher-order coherence functions. Importantly, the coherence functions can only be determined by the *Liouvillian eigenspace*.

We stress that when considering an NHH in an infinite-dimensional system, one can also determine infinitely-high-order HEPs related to high-order moments of the fields [64,65]. Nevertheless, NHHs fail to include quantum jumps and thermal noise. A striking example of this fact comes from an effective Hamiltonian which commutes with the total photon number operator. While at the NHH level this implies that manifolds with different numbers of excitations cannot interact, the quantum jumps of the Liouvillian can still mix states with different photon numbers. These different properties of the NHH and the corresponding Liouvillian imply that also their eigenstates are different [54,56]. As a result, the equations of motion for second- or higher-order

quantum field moments differ for the Liouvillian and NHH formalisms, indicating their distinguishable spectral properties. Hence, the analyses of higher-order HEPs resulting from higher-order moments are predictive only in the semiclassical regime, when the operators can be treated as c numbers. In other words, *a correct description of spectral properties of a quantum system via its higher-order field moments needs to rely exclusively on the Liouvillian eigenspectrum*.

D. Higher-order Liouvillian exceptional points in a bimodal cavity with incoherent mode coupling

As an example of the above general result, we study a bosonic system of a bimodal cavity with incoherent mode coupling to reveal its higher-order Liouvillian EPs. The incoherent mode interaction can be encoded by the off-diagonal elements of the damping matrix in the quantum Liouvillian [66,67]. These off-diagonal damping coefficients naturally appear in the microscopic theory of overlapping modes in open resonators [66] and chaotic two-mode lasers [68], where a strong interaction with the surrounding environment can induce a mode overlapping in the multimode cavities. Moreover, these theories have proved useful in explaining the Petermann excess noise factor in random lasers [68,69] and describing intensity-fluctuation spectra in bimodal cavities coupled to quantum emitters [70,71]. Interestingly, one of the first experimental observations of the intermode coupling in a bimodal cavity due to the interaction with surrounding screening fields, induced by a conducting sample located near the cavity, was already reported a few decades ago in Ref. [72], where microwave Hall measurements were performed. Incoherent mode coupling can be produced in various ways. These have been already realized, e.g., in anti- \mathcal{PT} -symmetrical *classical* systems, which include: nonlinear Brillouin scattering in a single microcavity [73], two passive waveguides, separated by a metallic film [74], counterpropagating media with heat exchange [75] or resistively coupled electric resonators [76], and, in quantum systems, through a coherent transport of flying atoms [77]. Nevertheless, in all these works, when studying EPs, a phenomenological approach has been utilized based exclusively on effective NHHs, thus ignoring quantum-jump effects.

In particular, we analyze second- and third-order LEPs of such systems arising from a HEP of second order by calculating the first- and second-order coherence functions, respectively. Also, we calculate the corresponding power and intensity-fluctuation spectra to reveal their squared and cubic Lorentzian expressions, accordingly. We also reveal the anti- \mathcal{PT} and \mathcal{PT} symmetries of the NHH, which connect the presence of LEPs with spontaneous breaking of such symmetries in a bimodal cavity with incoherent mode coupling. We show the possibility of switching between the \mathcal{PT} and anti- \mathcal{PT} symmetries of bosonic linear systems, like linearly coupled harmonic cavities. This switching can simply be realized by applying a tunable linear coupler (e.g., a tunable beam splitter in optical implementations) to a two-mode output field of the system. Moreover, our analysis of such two-mode systems reveals that, for a LEP of odd order $(2k+1)$, the system-enhanced sensitivity to external perturbations ϵ scales at most as $\epsilon^{\frac{1}{2k}}$.

We stress that the discussed \mathcal{PT} and anti- \mathcal{PT} symmetries in our study are exclusively related to the symmetries of the NHH, which plays a central role in our work. However, we note that the recent studies in Refs. [55,59,62] have already addressed the properties of \mathcal{PT} symmetry of the whole Liouvillian. Since we focus here on dissipative systems, the whole Liouvillian does not possess the \mathcal{PT} symmetry in the sense of, e.g., Ref. [20]. Nonetheless, the studied Liouvillian, for a two-site system, is *passively* \mathcal{PT} symmetric. In other words, it acquires the \mathcal{PT} symmetry in a reference frame with global decay [55,59]; that is, after applying an appropriate gauge transformation.

Let us recall now the meaning of the \mathcal{PT} and anti- \mathcal{PT} symmetries of the NHH. In general, a system described by a Hamiltonian \hat{H} exhibits \mathcal{PT} (anti- \mathcal{PT}) symmetry if \hat{H} commutes (anticommutes) with the \mathcal{PT} operator. Here the parity operator \mathcal{P} transforms a position operator \hat{x} to $-\hat{x}$ and a momentum operator \hat{p} to $-\hat{p}$, while the time reversal operator \mathcal{T} transforms $\hat{x} \rightarrow \hat{x}$ and $\hat{p} \rightarrow -\hat{p}$, and performs complex conjugation $i \rightarrow -i$.

This paper is organized as follows: In Sec. II, we briefly introduce a general Liouvillian for a dissipative multimode bosonic system. In Sec. III, we present our main result; namely, that any HEP of a finite order implies the infinite order of the corresponding LEP. As an example, we study higher-order LEPs in a bimodal cavity with incoherent mode coupling in Sec. IV. In particular, we analyze second- and third-order LEPs by means of the first- and second-order coherence functions, respectively, along with the power- and intensity-fluctuation spectra to reveal their squared and cubic Lorentzian lineshapes at the corresponding LEPs. We also demonstrate that EPs in such systems can be directly associated with \mathcal{PT} - and anti- \mathcal{PT} -symmetry breaking. Conclusions are drawn in Sec. V.

II. LIOUVILLIAN OF A GENERAL DISSIPATIVE LINEAR BOSONIC SYSTEM

The dynamics of a density matrix $\hat{\rho}$ describing a quantum system interacting with its environment is governed by a completely positive trace-preserving (CPTP) map. In the limit of weak Markovian time-independent interactions, such a CPTP map is known as the Liouvillian superoperator \mathcal{L} whose action is described by the master equation:

$$\frac{d}{dt}\hat{\rho}(t) = \mathcal{L}\hat{\rho}(t). \quad (1)$$

For an N -mode open linear coupled bosonic system interacting with thermal environment, the Liouvillian has the following general Gorini-Kossakowski-Sudarshan-Lindblad form ($\hbar = 1$) [66,67]:

$$\begin{aligned} \mathcal{L}\hat{\rho} = & -i[\hat{H}, \hat{\rho}] + \frac{n_{\text{th}} + 1}{2} \sum_{j,k}^N \gamma_{jk} \mathcal{D}[\hat{a}_j, \hat{a}_k^\dagger] \hat{\rho} \\ & + \frac{n_{\text{th}}}{2} \sum_{j,k}^N \gamma_{jk} \mathcal{D}[\hat{a}_j^\dagger, \hat{a}_k] \hat{\rho}, \end{aligned} \quad (2)$$

where \hat{H} is a Hermitian Hamiltonian, and the general Lindblad dissipators are

$$\mathcal{D}[\hat{\Gamma}_j, \hat{\Gamma}_k^\dagger] \hat{\rho} = 2\hat{\Gamma}_j \hat{\rho} \hat{\Gamma}_k^\dagger - \hat{\Gamma}_k^\dagger \hat{\Gamma}_j \hat{\rho} - \hat{\rho} \hat{\Gamma}_k^\dagger \hat{\Gamma}_j. \quad (3)$$

In Eq. (2), \hat{a}_j (\hat{a}_j^\dagger) is the annihilation (creation) operator of mode j ; the diagonal damping coefficients γ_{kk} denote the inner k th mode decay rate, while the off-diagonal coefficients γ_{jk} denote the *incoherent* coupling between modes j and k , due to the interaction of both modes with the environment [66]. Without loss of generality, we assume that the thermal photon number n_{th} is constant throughout the spectral range of a system. The Liouvillian \mathcal{L} can also be recast in the following form:

$$\mathcal{L}\hat{\rho} = -i(\hat{H}_{\text{eff}}\hat{\rho} - \hat{\rho}\hat{H}_{\text{eff}}^\dagger) + \frac{n_{\text{th}} + 1}{2} \sum_{j,k} \hat{a}_j \hat{\rho} \hat{a}_k^\dagger + \frac{n_{\text{th}}}{2} \sum_{j,k} \hat{a}_j^\dagger \hat{\rho} \hat{a}_k, \quad (4)$$

where \hat{H}_{eff} is an effective NHH given by

$$\hat{H}_{\text{eff}} = \hat{H} - \frac{i}{2} \sum_{j,k} \gamma_{jk} \hat{a}_j^\dagger \hat{a}_k. \quad (5)$$

Note that the term $(\hat{H}_{\text{eff}}\hat{\rho} - \hat{\rho}\hat{H}_{\text{eff}}^\dagger)$ in Eq. (4) can be interpreted as a generalized commutator. Moreover, this NHH is not Hermitian, i.e., $\hat{H}_{\text{eff}} \neq \hat{H}_{\text{eff}}^\dagger$. Additionally, the Hermitian Hamiltonian in Eq. (5) for a linear coupled system can be written in a general form

$$\hat{H} = \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k + \sum_{j < k} (\chi_{jk} \hat{a}_j^\dagger \hat{a}_k + \text{H.c.}), \quad (6)$$

where ω_k is a bare frequency of the mode k , and χ_{jk} is the coherent coupling coefficient between modes j and k .

III. LIOUVILLIAN EXCEPTIONAL POINTS OF ANY ORDER IN DISSIPATIVE LINEAR BOSONIC SYSTEMS

In a recent work [61], it has been shown that, if an NHH has an EP of n th order, then it implies that a LEP is at least of order $(2n - 1)$. Below, we demonstrate in a simple manner that if an NHH of a linear bosonic system has an EP of any order $n \geq 2$, determined by the first-order field moments, then this EP actually implies an infinite order for the LEP, which, accordingly, is determined by higher-order field moments, and, thus, can be identified by higher-order coherence functions.

To show this general result, let us first start describing the time dynamics of the system field averages $\langle \hat{a}_j(t) \rangle$. Note that we are working exclusively in the Schrödinger picture; thus, for simplicity, we put the time parameter t inside triangular brackets. After applying the formula for the time derivative of the field averages

$$\frac{d}{dt} \langle \hat{a}_j(t) \rangle = \text{Tr} \left[\hat{a}_j \frac{d}{dt} \hat{\rho}(t) \right], \quad (7)$$

and using Eqs. (1), (2), and (7), one obtains a linear system

$$v(t) = \exp(-i\hat{H}_{\text{eff}}t)v(0), \quad (8)$$

where $v(t) = [\langle \hat{a}_1(t) \rangle, \langle \hat{a}_2(t) \rangle, \dots, \langle \hat{a}_N(t) \rangle]^T$ is a vector of the operator averages, and \hat{H}_{eff} is a matrix form of the effective NHH in Eq. (5).

Hence, the dynamics of the annihilation operators imposed by the Liouvillian \mathcal{L} can be fully determined by the eigenspectrum of the effective NHH \hat{H}_{eff} . As a result, the appearance of an EP in the NHH spectrum immediately implies the emergence of the same EP in the Liouvillian spectrum [56]. Namely, the relationship between the eigenfrequencies (eigenvalues) $\nu \equiv \nu_{\text{NHH}}$ of the NHH and $\lambda \equiv \lambda_{\mathcal{L}}$ of the Liouvillian, which define the time dynamics of the fields in Eq. (8), bear a simple form [56]:

$$\lambda_{\mathcal{L}} = -i\nu_{\text{NHH}}. \quad (9)$$

Therefore, the coalescence of the eigenvalues ν_{NHH} of the NHH, along with its eigenstates, indicates the coalescence of the eigenvalues $\lambda_{\mathcal{L}}$ and the corresponding eigenstates of the Liouvillian. These eigenvalues merging cause the NHH to acquire a nondiagonal Jordan form, which is then reflected in the nonexponential character of the time evolution of the cavity fields according to Eq. (8). Moreover, the symmetry shared by the NHH becomes, in general, a local symmetry of the Liouvillian. The latter stems from the fact that the Liouvillian does not necessarily have the global symmetries of the NHH. We stress that the above conclusion of the coincidence of EPs of the NHH and Liouvillian is valid for any linear quadratic NHH in Eq. (5), with its coherent part given in Eq. (6). Moreover, the damping coefficients in the Lindblad dissipators in Eq. (3) may attain any form, e.g., similar to that of the Scully Lamb laser model [56], as long as the system remains linear, i.e., dissipative.

Higher-order correlation functions

The exact equivalence between the effective Hamiltonian and Liouvillian predictions of the spectral properties of the system holds true only for the dynamics of the annihilation operators. For example, higher-power field averages $\langle \hat{a}_j^{\dagger m} \hat{a}_j^n(t) \rangle$ would be affected by the presence of quantum jumps in a nontrivial way. Nevertheless, from the presence of a first-order HEP one we can deduce the properties of higher-order correlation functions, which, for linear systems, are determined by the higher-order field moments. Indeed, this function, at the steady state, can be calculated according to the formula

$$g_{j,ss}^{(1)}(\tau) = \frac{\langle \hat{a}_j^{\dagger}(0) \hat{a}_j(\tau) \rangle_{ss}}{\langle \hat{a}_j^{\dagger}(0) \hat{a}_j(0) \rangle_{ss}}, \quad j = 1, \dots, N, \quad (10)$$

where $\langle \hat{a}_j^{\dagger}(0) \hat{a}_j(\tau) \rangle_{ss}$ is a two-time correlation function (TTCF) for the mode j at the steady state. The TTCF, in turn, can be easily computed by exploiting the quantum regression theorem [63,78]. Namely, by solving the equations of motion for the field averages $\langle \hat{a}(\tau) \rangle$ in Eq. (8), one can immediately obtain $\langle \hat{a}^{\dagger}(0) \hat{a}(\tau) \rangle$. The TTCF for, e.g., the field \hat{a}_1 reads [56]

$$f_j(\tau) = \exp(-i\hat{H}_{\text{eff}}\tau) f_j(0), \quad (11)$$

where $f(\tau) = [\langle \hat{a}_1^{\dagger}(0) \hat{a}_1(\tau) \rangle_{ss}, \dots, \langle \hat{a}_N^{\dagger}(0) \hat{a}_N(\tau) \rangle_{ss}]^T$ is a vector of TTCFs for the mode j .

In other words, the dynamics and symmetry of the equations of motion for the coherence function $g^{(1)}(\tau)$ is determined by the same effective NHH. Again, at the HEP of n th order, the NHH obtains a Jordan form. As a result, and according to Eq. (11), the coherence function $g^{(1)}(\tau)$,

regardless of the mode j , attains a nonexponential form, with the highest power degree in τ , as follows:

$$g_{ss}^{(1)}(\tau) \sim \tau^{n-1} \exp(-i\nu_{\text{HEP}}\tau), \quad (12)$$

where ν_{HEP} , with imaginary part $\text{Im}(\nu_{\text{HEP}}) < 0$, is a complex eigenvalue of the NHH at a HEP of n th order.

The steady state of the considered linear systems with the Liouvillian \mathcal{L} , given in Eq. (2), along with the Hamiltonian in Eq. (6), is a thermal state. As a result, the higher-order coherence functions $g_{ss}^{(k)}(\tau)$, $k \in \mathbb{N}$, at the steady state, are completely determined by the first-order coherence function $g_{ss}^{(1)}(\tau)$, according to the moments theorem [63].

The higher-order coherence functions $g_{j,ss}^{(2k)}(\tau)$, based on the TTCFs, are found as

$$g_{j,ss}^{(2k)}(\tau) = \frac{\langle \hat{a}_j^{\dagger k}(0) \hat{a}_j^{\dagger k}(\tau) \hat{a}_j^k(\tau) \hat{a}_j^k(0) \rangle_{ss}}{\langle \hat{a}_j^{\dagger}(0) \hat{a}_j(0) \rangle_{ss}^{2k}}. \quad (13)$$

The form of the coherence function $g_{j,ss}^{(2k)}(\tau)$ in Eq. (13) ensures that at the n th-order HEP, the $(2k)$ th-order coherence function at the steady state contains the following term with the highest power degree in τ :

$$g_{ss}^{(2k)}(\tau) \sim \tau^{2k(n-1)} \exp[2k\text{Im}(\nu_{\text{HEP}})\tau]. \quad (14)$$

For instance, the second-order coherence function $g_{ss}^{(2)}(\tau)$ for the thermal light takes a simple form

$$g_{ss}^{(2)}(\tau) = 1 + |g_{ss}^{(1)}(\tau)|^2 \sim \tau^{2n-2} \exp[2\text{Im}(\nu_{\text{HEP}})\tau]. \quad (15)$$

The coherence function $g_{ss}^{(2k)}(\tau)$ is solely defined by the Liouvillian eigenspectrum; thus, at the HEP of the n th order, this function implies the coalescence of $[2k(n-1) + 1]$ Liouvillian eigenvectors. This means that the LEP becomes at least of the order $[2k(n-1) + 1]$. Moreover, because of the infinite-dimensional Hilbert space of a general bosonic system, the coherence function can be, thus, of infinite order. Thus, a HEP would simply imply the existence of the LEP of *infinite* order. In other words, the order of a LEP is only limited by the maximal possible number of photons in a system, i.e., the maximal size of its Hilbert space.

To shed more light on this direct connection between the higher-order LEPs and higher-order coherence functions, let us recall the general formula for the TTCFs for the steady state, which is used in the definition of the coherence function:

$$\langle \hat{\mathcal{O}}_1(0) \hat{\mathcal{O}}_2(\tau) \hat{\mathcal{O}}_3(0) \rangle_{ss} = \text{Tr}\{\hat{\mathcal{O}}_2(0) e^{\mathcal{L}\tau} [\hat{\mathcal{O}}_3(0) \hat{\rho}_{ss} \hat{\mathcal{O}}_1(0)]\}, \quad (16)$$

where $\hat{\mathcal{O}}_j$ are some system operators. The operator $\hat{\mathcal{O}}_3 \hat{\rho}_{ss} \hat{\mathcal{O}}_1$ in Eq. (16) leads the steady state $\hat{\rho}_{ss}$ into the new state, which becomes a decomposition of the Liouvillian eigenmatrices $\hat{\rho}_i$, i.e.,

$$\hat{\mathcal{O}}_3 \hat{\rho}_{ss} \hat{\mathcal{O}}_1 = \sum_i c_i \hat{\rho}_i. \quad (17)$$

By recalling the linearity of the trace, we have

$$\langle \hat{\mathcal{O}}_1(0) \hat{\mathcal{O}}_2(\tau) \hat{\mathcal{O}}_3(0) \rangle_{ss} = \sum_i c_i \text{Tr}\{\hat{\mathcal{O}}_2(0) e^{\mathcal{L}\tau} [\hat{\rho}_i]\}. \quad (18)$$

In the presence of a LEP, one finally has

$$\langle \hat{\mathcal{O}}_1(0) \hat{\mathcal{O}}_2(\tau) \hat{\mathcal{O}}_3(0) \rangle_{ss} = \sum_i c_i \tau^{n_i} e^{\lambda_i \tau} \text{Tr}\{\hat{\mathcal{O}}_2(0) \hat{\rho}_i\}, \quad (19)$$

where n_i is the degree of the degeneracy of the LEP associated with the eigenmatrix $\hat{\rho}_i$ (see also Ref. [56] for more details). Moreover, the eigenmatrices $\hat{\rho}_i$ correspond to various powers of the boson operators of the fields [79], which means that this is the eigenspace of higher-order field moments that is identified by the TTCFs. In our particular case, $\hat{O}_1 = \hat{a}^{\dagger k}$, $\hat{O}_2 = \hat{a}^{\dagger k} \hat{a}^k$, and $\hat{O}_3 = \hat{a}^k$. Note that TTCFs, in Eq. (16), can be applied to any moments of the field, i.e., not necessarily to the Hermitian moments $O_2(\tau) = \hat{a}^{\dagger k}(\tau) \hat{a}^k(\tau)$. As such, it is possible to reveal an arbitrary order m of a LEP, apart from that identified by the coherence function $g_{ss}^{(2k)}(\tau)$.

Experimentally, the spectral properties of dissipative systems have been measured, in particular regarding the closure of the Liouvillian gap occurring in dissipative phase transitions [80]. In particular, in Ref. [81] the two-time correlation function has been used to prove the occurrence of the first-order phase transition of a semiconductor micropillar, as predicted in Refs. [82,83]. Moreover, optical hysteresis properties have been used in Ref. [84] to prove the emergence of a critical slowing-down effect. Similarly, in Ref. [85] the emergence of a slow timescale in a one-dimensional superconductor chain has been used to pinpoint the precursors of a dissipative phase transition in a driven-dissipative Bose-Hubbard model [86,87].

We conclude that observing LEPs using the first- and second-order correlation functions is within the experimental reach of current techniques. For instance, an n th-order HEP would imply the possibility to measure a LEP at least of order $(2n - 1)$ [61] by means of the coherence function $g_{ss}^{(2)}(\tau)$. We note that, however, to access higher-order LEPs using higher-order coherence functions could be much more challenging. Indeed, $g_{ss}^{(2k)}(\tau)$ decays much faster than $g_{ss}^{(2)}(\tau)$ [cf., Eq. (14)], thus reducing the visibility of its nonexponential behavior.

IV. EXAMPLE OF A BIMODAL CAVITY WITH INCOHERENT MODE COUPLING

In this section, we study higher-order LEPs in a bimodal cavity with incoherent mode coupling. In particular, we analyze second- and third-order LEPs arising from a HEP of the second order by calculating first- and second-order coherence functions, respectively. Also, we calculate the corresponding power and intensity-fluctuation spectra to reveal their squared and cubic Lorentzian lineshapes, accordingly. Additionally, we reveal the anti- \mathcal{PT} and \mathcal{PT} symmetries of the NHH, which connect the presence of the LEPs with spontaneous breaking of such symmetries in the system.

A. Non-Hermitian Hamiltonian exceptional second-order points and its anti- \mathcal{PT} and \mathcal{PT} symmetries

1. Anti- \mathcal{PT} symmetry and exceptional point of an effective non-Hermitian Hamiltonian

The dynamics of a density matrix $\hat{\rho}$ of a bimodal cavity with incoherent mode coupling is described by the Liouvillian in Eq. (2) with the free coherent Hamiltonian of the form

$$\hat{H} = \sum_{k=1,2} \omega_k \hat{a}_k^{\dagger} \hat{a}_k. \quad (20)$$

For simplicity, we further assume that the damping matrix γ_{jk} is symmetric, i.e., $\gamma_{21} = \gamma_{12}$, and the inner mode decaying rates are the same, i.e., $\gamma_{11} = \gamma_{22} = \gamma$.

By working in the rotating reference frame with the central frequency $\bar{\omega} = (\omega_1 + \omega_2)/2$, the effective NHH, given in Eq. (5), attains the form

$$\hat{H}_{\text{eff}} = \frac{1}{2} \begin{pmatrix} \Delta - i\gamma & -i\gamma_{12} \\ -i\gamma_{12} & -\Delta - i\gamma \end{pmatrix}, \quad (21)$$

where $\Delta = (\omega_1 - \omega_2)$ is a cavity resonance difference.

This NHH is anti- \mathcal{PT} symmetric, meaning that its anti-commutator with a \mathcal{PT} operator is zero:

$$\mathcal{PT} \hat{H}_{\text{eff}} (\mathcal{PT})^{-1} = -\hat{H}_{\text{eff}}. \quad (22)$$

The action of the time-reversal operator \mathcal{T} on the NHH \hat{H}_{eff} is equivalent to

$$\mathcal{T} \hat{H}_{\text{eff}} \mathcal{T}^{-1} = \hat{H}_{\text{eff}}^*,$$

where the asterisk indicates complex conjugation. And the parity operator \mathcal{P} is equivalent to the Pauli $\hat{\sigma}_x$ matrix. The presence of the incoherent mode coupling rate γ_{12} , thus, induces the anti- \mathcal{PT} symmetry in the evolution of the field averages.

The eigenspectrum of the NHH gives two possible eigenvalues:

$$\nu_{1,2} = -i\frac{\gamma}{2} \pm \frac{i}{2}D, \quad (23)$$

where

$$D = \sqrt{\gamma_{12}^2 - \Delta^2}. \quad (24)$$

The unnormalized eigenvectors of the NHH, then, can be easily found as

$$\psi_{1,2} \equiv \begin{pmatrix} i\gamma_{12} \\ \Delta \mp iD \end{pmatrix}. \quad (25)$$

From Eqs. (23) and (25) it follows that the NHH \hat{H}_{eff} , and, thus, the Liouvillian, attain an EP

$$\gamma_{12}^{\text{EP}} = |\Delta| = |\omega_1 - \omega_2|. \quad (26)$$

At this EP, the NHH experiences a spectral-phase transition, associated with anti- \mathcal{PT} -symmetry breaking, meaning that cavity fields in Eq. (8) can exist in two different spectral phases.

We plot the eigenfrequencies ν_1 and ν_2 in Fig. 1. When $\gamma_{12} > \Delta$, i.e., the NHH eigenvalues ν_1 and ν_2 both become purely imaginary, the NHH eigenstates ψ_1 and ψ_2 are in the exact anti- \mathcal{PT} -symmetric phase. That is, the eigenmodes ψ_1 and ψ_2 become asymmetric, the former attaining an effective gain and the latter acquiring additional losses proportional to the parameter $|D|$ [see Eq. (23)]. Contrary to this, when $\gamma_{12} < \Delta$, the eigenvalues are no longer purely imaginary, and the eigenstates of the NHH are in the broken anti- \mathcal{PT} -symmetric phase. In this case, the two modes are spectrally separated, and this mode splitting can be defined by the real-valued parameter $|D|$ given in Eq. (23).

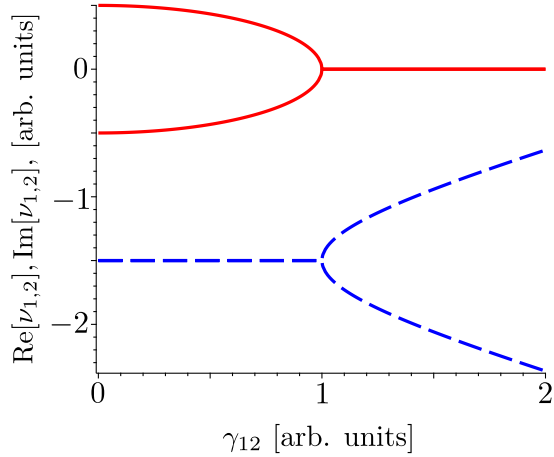


FIG. 1. Real (red solid curves) and imaginary (blue dashed curves) parts of the eigenfrequencies ν_1 and ν_2 of the effective NHH \hat{H}_{eff} versus the incoherent coupling rate γ_{12} . These eigenfrequencies constitute a subset of the eigenfrequencies of the Liouvillian \mathcal{L} , of a bimodal cavity according to Eq. (23). The chosen system parameters are the frequencies of the modes $\omega_2 = -\omega_1 = 0.5$ [arbitrary units]; the cavity losses for both modes \hat{a}_1 and \hat{a}_2 are $\gamma = 3$ [arb. units]. The system experiences a spectral-phase transition at the EP γ_{12}^{EP1} , according to Eq. (26), due to the interplay between the mode frequency difference Δ and the incoherent mode coupling γ_{12} .

2. Switching to the \mathcal{PT} -symmetric modes

By appropriate unitary transformations, the effective NHH in Eq. (21) can be recast into a form, where it acquires a \mathcal{PT} -like symmetry, namely, *passive* \mathcal{PT} symmetry. For this, we can introduce the general *combined modes* (often referred to as *supermodes*) \hat{c}_1 and \hat{c}_2 (see, e.g., Refs. [70,88]), defined via the rotation

$$\begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \quad (27)$$

where θ is an appropriate angle. This transformation of \hat{a}_1 and \hat{a}_2 to the supermodes \hat{c}_1 and \hat{c}_2 can simply be realized with a tunable linear coupler. In case of optical implementations of our system, this coupler can be realized by a single tunable beam splitter or, in more refined implementations, by a Mach-Zehnder interferometer [89].

For the case under consideration, i.e., $\gamma_{12} = \gamma_{21}$, the Lindblad master equation, given in Eq. (2), can be put in the diagonal form of the damping matrix γ_{jk} by considering $\theta = \pi/4$, i.e., $\hat{c}_{1,2} = (\hat{a}_1 \pm \hat{a}_2)/\sqrt{2}$. We have

$$\begin{aligned} \mathcal{L}\hat{\rho} = & -i[\hat{H}_{c_1,c_2}, \hat{\rho}] + \sum_{k=1,2} \frac{\gamma_{c_k}(n_{\text{th}} + 1)}{2} \mathcal{D}[\hat{c}_k]\hat{\rho} \\ & + \frac{\gamma_{c_k}n_{\text{th}}}{2} \mathcal{D}[\hat{c}_k^\dagger]\hat{\rho}, \end{aligned} \quad (28)$$

where

$$\hat{H}_{c_1,c_2} = \sum_k \bar{\omega} \hat{c}_k^\dagger \hat{c}_k + \frac{\Delta}{2} (\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1), \quad (29)$$

where $\gamma_{c_1,c_2} = \gamma \mp \gamma_{12}$. Hence, the model of the two incoherently coupled modes \hat{a}_1 and \hat{a}_2 becomes that of two dissipative

coherently coupled modes \hat{c}_1 and \hat{c}_2 in the appropriate basis (see Appendix) for a general form of the Liouvillian \mathcal{L} , under the transformation in Eq. (27).

The effective NHH \hat{H}_{eff} in Eq. (21), in a rotating reference frame $\bar{\omega}$, then reads

$$\hat{H}_{c_1,c_2}^{\text{eff}} = \frac{1}{2} \begin{pmatrix} -i\gamma_{c_1} & \Delta \\ \Delta & -i\gamma_{c_2} \end{pmatrix}. \quad (30)$$

This effective NHH in Eq. (30) now indicates that the supermodes \hat{c}_1 and \hat{c}_2 constitute a *passive* \mathcal{PT} -symmetric system [1,2,21,22]. Namely, if one applies a gauge transformation

$$\begin{pmatrix} \hat{c}'_1 \\ \hat{c}'_2 \end{pmatrix} = \exp\left(-\frac{\gamma}{2}t\right) \begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \end{pmatrix}, \quad (31)$$

the modified NHH in Eq. (30) then reads

$$\hat{H}_{c_1,c_2}'^{\text{eff}} = \frac{1}{2} \begin{pmatrix} i\gamma_{12} & \Delta \\ \Delta & -i\gamma_{12} \end{pmatrix}. \quad (32)$$

This NHH $\hat{H}_{c_1,c_2}'^{\text{eff}}$, in Eq. (32), commutes with the \mathcal{PT} operator, i.e.,

$$[\hat{H}_{c_1,c_2}'^{\text{eff}}, \mathcal{PT}] = 0. \quad (33)$$

In other words, *the initial loss-loss dynamics* for the supermodes \hat{c}_1 and \hat{c}_2 *becomes equivalent to the balanced gain-loss evolution*, apart from the global decay rate $\gamma/2$.

3. Summary

To sum up, we have shown that, by applying appropriate unitary transformations to the anti- \mathcal{PT} -symmetric NHH in Eq. (5), one can readily discover a hidden \mathcal{PT} symmetry of the NHH. As such, the EP in Eq. (26) is associated not only with the anti- \mathcal{PT} -symmetry but also with \mathcal{PT} -symmetry breaking of the NHH, induced by the same interplay between the frequency difference Δ and incoherent coupling rate γ_{12} .

Note that the exact (broken) anti- \mathcal{PT} -symmetric phase is accompanied by the broken (exact) \mathcal{PT} -symmetric phase. Indeed, the eigenfrequencies of the NHH are left unchanged under the unitary transformations in Eq. (27), but the action of the \mathcal{PT} and anti- \mathcal{PT} symmetry is opposite for the purely imaginary eigenfrequencies. Such coexistence of the opposite symmetric phases has already been pointed out in Refs. [73,90]. We stress that the possibility to witness different symmetries of the system by considering different operators offers great flexibility to explore different dynamical regimes and various kinds of nontrivial light behavior in this system in the semiclassical regime [73–75]. Moreover, we have explained that one can witness not only the \mathcal{PT} and anti- \mathcal{PT} symmetries, but can also physically switch between them by transforming the system two-mode output fields with an additional tunable linear coupler, e.g., a tunable beam splitter in optical implementations of the general model discussed.

B. Liouvillian exceptional points of higher orders

As it has been already stressed in Sec. III, *a HEP of any order n implies a LEP of any higher order $m \geq n$* . Below, we study the second- and third-order LEPs, which arise due to the presence of a second-order HEP in the system under consideration.

1. Liouvillian exceptional point of second order and squared Lorentzian power spectra

The presence of an EP of second order in Eq. (26) can be signalled by a squared Lorentzian lineshape in the power spectrum of the \mathcal{PT} -symmetric modes \hat{c}_1 and \hat{c}_2 [29,91–93]. Indeed, the anti- \mathcal{PT} -symmetric modes \hat{a}_1 and \hat{a}_2 generate two independent spectra, which are not coupled. Thus, it is impossible to see spectral lines merging which are highlighted by a squared Lorentzian for these modes. However, the spectra of the \mathcal{PT} -symmetric supermodes \hat{c}_1 and \hat{c}_2 can demonstrate the modes coalescence effect, characterized by the appearance of a *plateau* at the top of the lineshape curve.

The power spectrum $S^{(1)}(\omega)$ can be expressed via the first-order coherence function $g_{ss}^{(1)}(\tau)$ as

$$S^{(1)}(\omega) = \frac{1}{\pi} \text{Re} \int_0^{\infty} g_{ss}^{(1)}(\tau) \exp(i\omega\tau) d\tau. \quad (34)$$

Mathematically, $S^{(1)}(\omega)$ is, thus, the Fourier transform of the coherence function $g_{ss}^{(1)}(\tau)$, and, roughly speaking, indicates the response of the system to the injection of one particle at a frequency ω .

The coherence function $g_{ss}^{(1)}(\tau)$, for the supermodes \hat{c}_1 and \hat{c}_2 , with the help of Eq. (11), is found as

$$g_{\hat{c}_1, \hat{c}_2}^{(1)}(\tau) = \frac{\exp\left(-\frac{\gamma\tau}{2} - i\bar{\omega}\tau\right)}{D} \left(D \cosh \frac{D\tau}{2} \pm \gamma_{12} \sinh \frac{D\tau}{2} \right). \quad (35)$$

The incoherent power spectra of the supermodes \hat{c}_1 and \hat{c}_2 , thus, take the form

$$S_{\hat{c}_1, \hat{c}_2}^{(1)}(\omega) = \frac{1}{\pi D} \left(\frac{K_+(D \mp \gamma_{12})}{\Omega^2 + K_+^2} + \frac{K_-(D \pm \gamma_{12})}{\Omega^2 + K_-^2} \right), \quad (36)$$

where $\Omega = \omega - \bar{\omega}$, and $K_{\pm} = (\gamma \pm D)/2$.

In Fig. 2, we plot both the coherence function $g_{\hat{c}_1, \hat{c}_2}^{(1)}(\tau)$ and power spectra $S_{\hat{c}_1, \hat{c}_2}^{(1)}(\omega)$ for the supermodes \hat{c}_1 and \hat{c}_2 . Away from an EP, the coherence functions (power spectra) are a combination of two exponents (Lorentzians) for both fields \hat{c}_1 and \hat{c}_2 . When $\gamma_{12} = 0$, the functions $g_{\hat{c}_1, \hat{c}_2}^{(1)}(\tau)$ and spectra $S_{\hat{c}_1, \hat{c}_2}^{(1)}(\omega)$ are identical, and the system is in the exact (broken) \mathcal{PT} -symmetric (anti- \mathcal{PT} -symmetric) phase of the NHH. But if the losses γ are sufficiently large, the two cavity resonances ω_1 and ω_2 might not be resolved [see Fig. 2(d)]. At the EP $\gamma_{12} = \gamma_{12}^{\text{EP}}$, the coherence functions attain a nonexponential form:

$$g_{\hat{c}_1, \hat{c}_2}^{(1)}(\tau) = \exp\left(-\frac{\gamma\tau}{2} - i\bar{\omega}\tau\right) \left(1 \pm \frac{\gamma_{12}\tau}{2} \right). \quad (37)$$

We note that distinguishing the exponential from nonexponential behavior could be hard in practice [see Fig. 2(b)]. On the other hand, at the EP, the two spectra become a combination of the Lorentzian and squared Lorentzian lineshapes [see Fig. 2(e)]. Indeed, the power spectra $S_{\hat{c}_1, \hat{c}_2}^{(1)}$ at the EP become

$$S_{\hat{c}_1, \hat{c}_2}^{\text{EP}_1}(\omega) = \frac{1}{\pi} \frac{4}{\gamma^2 + 4\Omega^2} \left(\gamma \mp \gamma_{12} \pm \frac{2\gamma^2\gamma_{12}}{\gamma^2 + 4\Omega^2} \right), \quad (38)$$

where Ω is given in Eq. (36).

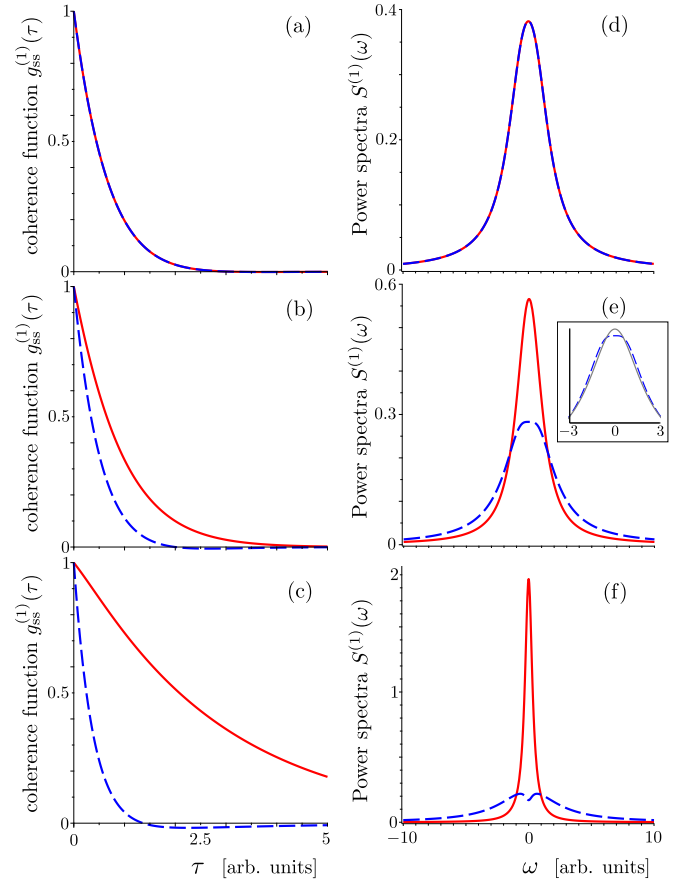


FIG. 2. First-order coherence function $g_{ss}^{(1)}(\tau)$ [panels (a)–(c)] and power spectra $S^{(1)}(\omega)$ [panels (d)–(f)] of the supermodes \hat{c}_1 (red solid curves) and \hat{c}_2 (blue dashed curves), according to Eqs. (35) and (36), respectively, of the bimodal cavity for various values of incoherent mode coupling rate γ_{12} : (a), (d) $\gamma_{12} = 0$ [arb. units]; (b), (e) $\gamma_{12} = \gamma_{12}^{\text{EP}_1} = 1$ [arb. units]; and (c), (f) $\gamma_{12} = 2$ [arb. units]. The reference frame is rotating at the central cavity frequency $\bar{\omega}$. The remaining system parameters are the same as in Fig. 1. The spectra of the supermodes \hat{c}_1 and \hat{c}_2 reveal the squared Lorentzian lineshape at the second-order EP, characterized by a *plateau* at the top of the curve [see panel (e), blue dashed curve]. The inset in panel (e) shows the best Lorentzian fit, represented by a sum of two Lorentzians (gray solid curve) of the spectrum at the EP (blue dashed curve), which highlights the distinctiveness of the squared Lorentzian. Thus, one can experimentally witness the presence of the EP by studying the power spectra for the model under consideration. Moreover, with increasing values of the incoherent mode-coupling rate $\gamma_{12} > \gamma_{12}^{\text{EP}_1}$, the supermode \hat{c}_2 acquires a volcanic cone shape (blue dashed curve), indicating that its spectrum is represented by the difference of two Lorentzians [panel (f)].

For larger values of $\gamma_{12} > \gamma_{12}^{\text{EP}}$, the first mode \hat{c}_1 experiences further amplification, whereas the second mode \hat{c}_2 encounters an increased damping. Notice also that the \hat{c}_2 mode, representing the difference of two Lorentzians, is no longer characterized by a single maximum for sufficiently large γ_{12} [see Fig. 2(c)]. The Lorentzian subtraction can lead to a substantial decrease of the spectral signal at the central cavity frequency $\bar{\omega}$, meaning that the energy is completely transferred from the mode \hat{c}_2 to \hat{c}_1 . The latter leads to the

observation of electromagnetically induced absorption in the system [73]. Clearly, the presence of the incoherent mode coupling gives rise to a number of nontrivial phenomena in the system, which are related to both anti- \mathcal{PT} - and \mathcal{PT} -symmetric systems.

2. Liouvillian exceptional point of third order and cubic Lorentzian intensity-fluctuation spectra

According to Eq. (15), the second-order coherence function $g^{(2)}(\tau)$ at the steady state is determined by the coherence function $g^{(1)}(\tau)$. By exploiting Eqs. (15) and (35), one can arrive to an explicit expression for the function $g_{\hat{c}_1, \hat{c}_2}^{(2)}(\tau)$. Namely, at the EP in Eq. (26), from Eq. (37), one obtains

$$g_{\hat{c}_1, \hat{c}_2}^{(2)}(\tau) = 1 + \exp(-\gamma\tau) \left(1 \pm \frac{\gamma_{12}\tau}{2}\right)^2, \quad (39)$$

meaning that the function $g_{\hat{c}_1, \hat{c}_2}^{(2)}(\tau)$ signals the presence of a LEP of the third order. This third-order LEP is also reflected by the cubic Lorentzian lineshape in the intensity-fluctuation spectra, which is defined by the second-order coherence function $g^{(2)}(\tau)$ at the steady state as follows:

$$S^{(2)}(\omega) = \frac{1}{\pi} \text{Re} \int_0^{\infty} [g^{(2)}(\tau) - 1] \exp(i\omega\tau) d\tau. \quad (40)$$

By combining Eqs. (39) and (40), one easily arrives at the following expressions for the intensity-fluctuation spectra (or noise spectra) for the supermodes at the EP:

$$S_{\hat{c}_1, \hat{c}_2}^{(2)}(\omega) \equiv \frac{\gamma \mp \gamma_{12}}{\omega^2 + \gamma^2} - \frac{\gamma \gamma_{12} (3\gamma_{12} \mp 4\gamma)}{2(\omega^2 + \gamma^2)^2} + \frac{2\gamma^3 \gamma_{12}^2}{(\omega^2 + \gamma^2)^3}. \quad (41)$$

As predicted by Eq. (41), at the EP $\gamma = \gamma_{12}^{\text{EP}}$ of Eq. (26) the intensity-fluctuation spectra are cubic Lorentzians [see Figs. 3(d)–3(f)]. Experimentally, resolving the exact cubic lineshape might be challenging due to the simultaneous presence of the squared and cubic Lorentzians terms in Eq. (41). Thus, other techniques and methods might be required to precisely detect it [56]. As Fig. 3 shows, the intensity-fluctuations decrease (increase) for the mode \hat{c}_1 (\hat{c}_2), in accordance with the power spectra in Fig. 2.

3. Liouvillian exceptional point of third order explicitly defined from the higher-order field moments matrix

From Eqs. (35) and (15) one can even explicitly identify the Liouvillian eigenvalues, which determine the second-order coherence function $g^{(2)}(\tau)$, and the merging of which gives rise to a third-order LEP. We have already shown in Fig. 3 how a third-order LEP gives rise to the cubic Lorentzian in the intensity-fluctuation spectra, along with a quadratic-time dynamics for the second-order coherence function $g^{(2)}(\tau)$ in Eq. (39). The system is $U(1)$ symmetric; that is, the Liouvillian is invariant under any phase shift ϕ of the boson operators $\hat{a}_j \rightarrow \hat{a}_j \exp(i\phi)$, $j = 1, 2$. Thus, the time dynamics of the coherence function $g^{(2)}(\tau)$ is completely determined by the second-order moments of the fields of the form $\langle \hat{a}_k^\dagger \hat{a}_l \rangle$, $k, l = 1, 2$. The dynamics of such moments is determined by the corresponding Liouvillian eigenspace. Therefore, the

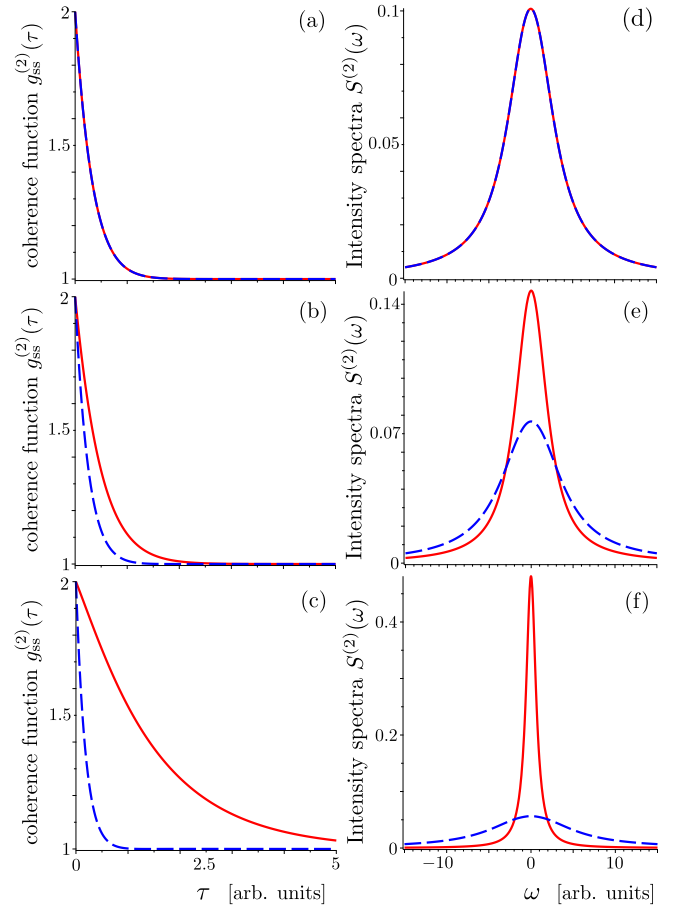


FIG. 3. Second-order coherence function $g_{\text{ss}}^{(2)}(\tau)$ [panels (a)–(c)] and intensity-fluctuation spectra $S^{(2)}(\omega)$ [panels (d)–(f)] of the supermodes \hat{c}_1 (red solid curves) and \hat{c}_2 (blue dashed curves), according to Eqs. (15) and (40), respectively, of the bimodal cavity for various values of incoherent mode coupling rate γ_{12} . The system parameters for each panel are the same as in Fig. 2.

knowledge of the time evolution of the second-order moments $\langle \hat{a}_k^\dagger \hat{a}_l \rangle$ can reveal the presence of the third-order LEP.

The dynamics of the second-order moments can be described by the averages of $\hat{\mathbf{a}} = [(\hat{a}_1^\dagger, \hat{a}_2^\dagger) \otimes (\hat{a}_1, \hat{a}_2)]^T$. Indeed,

$$\frac{d}{dt} \langle \hat{\mathbf{a}} \rangle = \mathbf{M} \langle \hat{\mathbf{a}} \rangle + n_{\text{th}} \mathbf{b}. \quad (42)$$

The evolution matrix \mathbf{M} for this vector of the averages $\langle \hat{\mathbf{a}} \rangle$ reads as

$$\mathbf{M} = \frac{1}{2} \begin{pmatrix} -2\gamma & -\gamma_{12} & -\gamma_{12} & 0 \\ -\gamma_{12} & 2i\Delta - 2\gamma & 0 & -\gamma_{12} \\ -\gamma_{12} & 0 & -2i\Delta - 2\gamma & -\gamma_{12} \\ 0 & -\gamma_{12} & -\gamma_{12} & -2\gamma \end{pmatrix}, \quad (43)$$

and the thermal noise vector is $\mathbf{b} = [\gamma, \gamma_{12}, \gamma_{12}, \gamma]^T$.

In the supermode basis, the evolution matrix \mathbf{N} and the noise vector \mathbf{d} for the vector $\hat{\mathbf{c}} = [(\hat{c}_1^\dagger, \hat{c}_2^\dagger) \otimes (\hat{c}_1, \hat{c}_2)]^T$ are easily found via the transformation

$$\mathbf{N} = \mathbf{TMT}^{-1}, \quad \mathbf{d} = \mathbf{Tb}, \quad (44)$$

where the 4×4 transformation matrix \mathbf{T} is given by

$$\mathbf{T} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \quad (45)$$

From Eqs. (42) and (44) it is evident that the dynamics for the vector of the averages $\langle \hat{\mathbf{a}} \rangle$ ($\langle \hat{\mathbf{c}} \rangle$) cannot, in general, possess anti- \mathcal{PT} (\mathcal{PT}) symmetry because of the presence of the thermal noise in the form of the vector \mathbf{b} (\mathbf{d}). Nevertheless, the matrix $i\mathbf{M}$ ($i\mathbf{N}$) is anti- \mathcal{PT} (\mathcal{PT}) symmetric. Indeed, the parity operator \mathcal{P} for the vector of the operators $\hat{\mathbf{a}}$, $\hat{\mathbf{c}}$ becomes

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (46)$$

With the help of Eq. (46) one can easily check that

$$\begin{aligned} \mathcal{PT}(i\mathbf{M})(\mathcal{PT})^{-1} &= -i\mathbf{M}, \\ \mathcal{PT}(i\mathbf{N})(\mathcal{PT})^{-1} &= i\mathbf{N}', \end{aligned} \quad (47)$$

where the modified matrix \mathbf{N}' is obtained from \mathbf{N} by applying the gauge transformation in Eq. (31). The inclusion of the imaginary prefactor i in the matrices \mathbf{M} and \mathbf{N}' in Eq. (47) ensures that the left-hand side of the equations of motion for the vectors of the operators $\langle \hat{\mathbf{a}} \rangle$ and $\langle \hat{\mathbf{c}} \rangle$ remain unchanged under \mathcal{PT} transformation. Thus, in the absence of thermal photons $n_{\text{th}} = 0$, the dynamics for the averaged vector of operators $\langle \hat{\mathbf{a}} \rangle$ ($\langle \hat{\mathbf{c}} \rangle$) restores the same anti- \mathcal{PT} (\mathcal{PT}) symmetry imposed by the effective NHH on quantum fields in Eq. (21) [(30)]. As a result, the LEP of third order, determined from the evolution matrix \mathbf{M} (\mathbf{N}), becomes directly associated with anti- \mathcal{PT} -symmetry (\mathcal{PT} -symmetry) breaking.

The eigenvalues of the matrix \mathbf{M} , and thus of the matrix \mathbf{N} , are found as follows:

$$\lambda_{1,2} = -2i\nu_{12}, \quad \lambda_{3,4} = -\gamma, \quad (48)$$

where ν_1 and ν_2 are the eigenvalues of the NHH given in Eq. (23). We plot these eigenvalues in Fig. 4. At the EP γ_{12}^{EP} , the algebraic multiplicity of the eigenvalue $\lambda = -\gamma$ equals four, whereas geometric multiplicity is three. In other words, there is a coalescence of three Liouvillian eigenvectors, which are determined by the moments of the operators in the vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$, but the derivation of their explicit form might require other approaches [79,94]. We also note that this finding of the LEP of third order for a second-order HEP in the space of the vector $\hat{\mathbf{c}}$ has already been observed in the single-photon regime for a similar system [56].

The presence of higher-order EPs in a system is usually associated with the enhanced system sensitivity to external perturbations in the vicinity of the EPs [1,4]. This system's enhanced spectral response $\Delta\omega$ near an EP of an n th-order to a perturbation ϵ scales as $\Delta\omega \sim \epsilon^{\frac{1}{n}}$. Remarkably, the system spectral sensitivity around the LEP of the third order can remain the same as it is near the second-order LEP. That is, the Liouvillian eigenvalues split near the third-order LEP, as $\Delta\lambda \sim \sqrt{\epsilon}$, not as a cubic root as one might expect. This square-root dependence on perturbation around the third-order LEP arises from the system symmetry and the nature of the applied perturbation.

As explained earlier, the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ [Eq. (48)] belong to the corresponding $U(1)$ Liouvillian eigenspace. As such, any perturbation of a single system parameter preserves

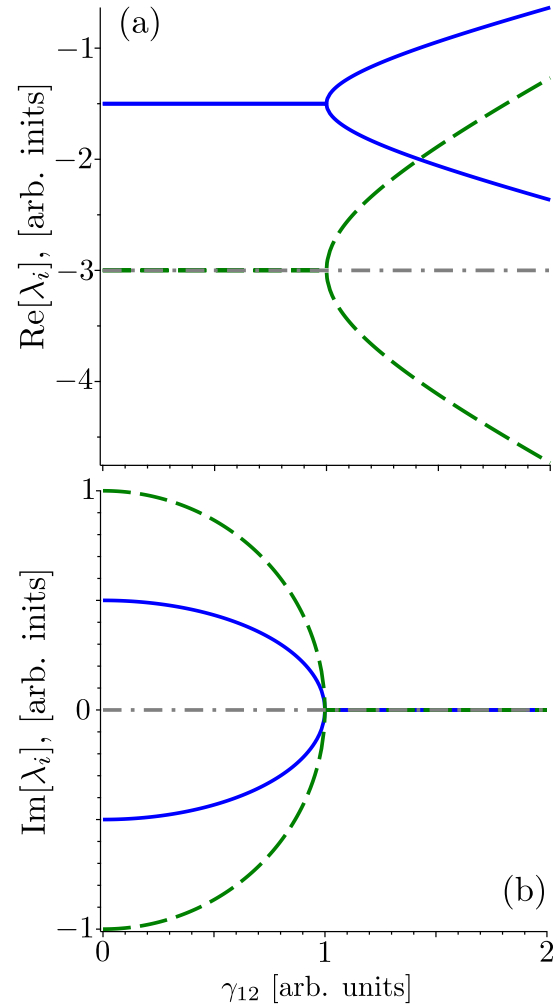


FIG. 4. (a) Real and (b) imaginary parts of the eigenvalues λ of the Liouvillian, determined from the evolution matrix \mathbf{M} : $\lambda_{1,2}$ (green dashed curves), and $\lambda_{3,4}$ (gray dash-dotted lines), according to Eq. (48), versus the incoherent coupling rate γ_{12} . The chosen system parameters are the same as in Fig. 1. For comparison, the Liouvillian eigenvalues defined by the NHH from Fig. 1 are shown as blue solid curves. Thus, for a given subspace of the Liouvillian eigenspace, determined by the NHH and evolution matrix \mathbf{M} , the EP γ^{EP} in Eq. (26) becomes a LEP of the second and third orders, respectively.

the $U(1)$ symmetry of the system, e.g., $\gamma \rightarrow \gamma + \epsilon$. Consequently, the eigenvalue λ_3 (and the corresponding Liouvillian eigenmatrix) remain real (and Hermitian) under such a perturbation. Since λ_3 never acquires an imaginary part for such perturbations, only the complex eigenvalues λ_1 and λ_2 , along with their eigenstates, induce a line splitting in the intensity-fluctuation spectrum around the third-order LEP. Thus, the spectral response to such external perturbations scales only as the square root at the third-order LEP.

This result can be generalized to higher-order LEPs. Since any Liouvillian eigenvalue comes in conjugate pairs, and given the \mathcal{PT} -symmetric structure of the system, given the coalescence of $(2k + 1)$ eigenvalues, one eigenvalue must always remain purely real in the vicinity of the LEP. Such

an eigenvalue cannot contribute to the enhanced system spectral response under a perturbation. We conclude that, for the studied system, for any LEPs of odd order ($2k + 1$), the system-enhanced sensitivity scales at most as $\epsilon^{\frac{1}{2k}}$ around the LEP.

Finally, we would like to stress the mentioned difference in the dynamics of the higher-order field moments imposed by the Liouvillian and NHH. To show this explicitly we write the corresponding evolution matrix \mathbf{M}_{NHH} for the vector of the operators $\hat{\mathbf{a}}$, derived from the NHH in Eq. (21) as follows:

$$\mathbf{M}_{\text{NHH}} = \frac{1}{2} \begin{pmatrix} 0 & \gamma_{12} & -\gamma_{12} & 0 \\ \gamma_{12} & 2i\Delta & 0 & -\gamma_{12} \\ -\gamma_{12} & 0 & -2i\Delta & \gamma_{12} \\ 0 & -\gamma_{12} & \gamma_{12} & 0 \end{pmatrix}. \quad (49)$$

A comparison of Eqs. (43) and (49) demonstrates that, indeed, the evolution imposed on the same operators is different in the Liouvillian and NHH formalisms. The eigenvalues of the matrix \mathbf{M}_{NHH} are similar to those in Eq. (48) but shifted by the value of γ , i.e., $\lambda_{\text{NHH}} = \lambda_{\mathcal{L}} + \gamma$, where $\lambda_{\mathcal{L}}$ are given in Eq. (48). Additionally, the inhomogeneous term \mathbf{b} , arising from the thermal noise, is absent in the equations of motion for the operators $\hat{\mathbf{a}}$ in the NHH formalism. Therefore, although the HEPs and LEPs can coincide for the same field moments, the system eigenspectra and dynamics are different in both formalisms [54,56].

V. CONCLUSIONS

In this work, we have demonstrated that, for a dissipative linear bosonic system whose effective NHH exhibits a HEP of any finite order n , determined by the first-order field moments, its corresponding LEPs can become at least of order $m \geq 2k(n - 1) + 1$, for $k = \frac{1}{2}, 1, 2, 3, \dots$, which can be determined by higher-order field moments, accordingly. These higher-order field moments are directly related to the *normally ordered* higher-order coherence functions via the quantum regression theorem.

Thus, we have shown how the coherence functions can offer a convenient tool to probe extreme system sensitivity to external perturbations in the vicinity of higher-order LEPs.

As an example, we have studied a bosonic system of a bimodal cavity with incoherent mode coupling to reveal its higher-order LEPs. In particular, we analyze second- and third-order LEPs of such systems arising from a HEP of second order by calculating the first- and second-order coherence functions, respectively. Also, we calculate the corresponding power and intensity-fluctuation spectra to reveal their squared and cubic Lorentzian expressions, accordingly. Moreover, our analysis of such two-mode systems indicates that, for the LEPs of an odd order ($2k + 1$), the system-enhanced sensitivity to external perturbations ϵ scales at most as $\epsilon^{\frac{1}{2k}}$.

We also reveal the anti- \mathcal{PT} and \mathcal{PT} symmetries of the NHH, which connect the presence of LEPs with spontaneous breaking of such symmetries in a bimodal cavity with incoherent mode coupling. Moreover, we showed the possibility of switching between the \mathcal{PT} and anti- \mathcal{PT} symmetries of the studied bosonic linear systems by applying an additional tunable linear coupler to two supermodes of the system output. By means of such a coupler one can, thus, transform the

initial loss-loss dynamics for the supermodes to the equivalent system with a balanced gain-loss evolution. In case of optical systems, this transformation can be implemented with a single tunable beam splitter.

We note that usually EPs have been studied in two- or multiparty systems with gain and loss (i.e., lossy driven systems). Here, we have analyzed EPs in a multiparty system without gain, but instead with its subsystems exhibiting losses with different rates. Such a system effectively leads to a model of a lossy-driven system.

In conclusion, we believe that our work has shown that the concept of quantum EPs, as defined via degeneracies of Liouvillians, is not only of a pure theoretical interest. We have demonstrated explicitly that the formalism of Ref. [54] can be tested experimentally at least for general quantum linear bosonic systems. In particular, LEPs can indeed be identified by measuring coherence functions or the power and intensity fluctuation spectra.

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APPENDIX: LIOUVILLIAN \mathcal{L} FOR SUPERMODES

Here we show an explicit form of the Liouvillian \mathcal{L} for the supermodes \hat{c}_1 and \hat{c}_2 in Eq. (27).

Considering a general Liouvillian, given in Eq. (2), for two modes with arbitrary damping matrix γ_{jk} and free Hermitian Hamiltonian in Eq. (20), after applying the transformations in Eq. (27), one arrives at the transformed master equation given by

$$\begin{aligned} \frac{d}{dt} \hat{\rho} = \mathcal{L} \hat{\rho} = & -i[\hat{H}_{c_1} + \hat{H}_{c_2} + \hat{H}_{c_1, c_2}, \hat{\rho}] \hat{\rho} \\ & + \mathcal{L}_{c_1} \hat{\rho} + \mathcal{L}_{c_2} + \mathcal{L}_{c_1, c_2} \hat{\rho} + \mathcal{L}_{c_2, c_1} \hat{\rho}, \end{aligned} \quad (\text{A1})$$

where the coherent part includes

$$\begin{aligned} \hat{H}_{c_1} &= (\omega_1 \cos^2 \theta + \omega_2 \sin^2 \theta) \hat{c}_1^\dagger \hat{c}_1, \\ \hat{H}_{c_2} &= (\omega_1 \sin^2 \theta + \omega_2 \cos^2 \theta) \hat{c}_2^\dagger \hat{c}_2, \\ \hat{H}_{c_1, c_2} &= \frac{1}{2} (\omega_1 - \omega_2) \sin 2\theta (\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1), \end{aligned} \quad (\text{A2})$$

and the incoherent Lindbladian part is

$$\mathcal{L}_{c_m}\hat{\rho} = \frac{n_{\text{th}} + 1}{2}A_m\mathcal{D}[\hat{c}_m] + \frac{n_{\text{th}}}{2}A_m\mathcal{D}[\hat{c}_m^\dagger], \quad (\text{A3})$$

for $m = 1, 2$, and

$$\mathcal{L}_{c_j, c_k}\hat{\rho} = \frac{n_{\text{th}} + 1}{2}A_{jk}\mathcal{D}[\hat{c}_j\hat{c}_k^\dagger] + \frac{n_{\text{th}}}{2}A_{jk}\mathcal{D}[\hat{c}_j^\dagger\hat{c}_k], \quad (\text{A4})$$

with $j, k = 1, 2$ and $j \neq k$. Moreover we have denoted

$$A_1 = \gamma_{11}\cos^2\theta + \gamma_{22}\sin^2\theta - \bar{\gamma}_{12}\sin 2\theta,$$

$$A_2 = \gamma_{11}\sin^2\theta + \gamma_{22}\cos^2\theta + \bar{\gamma}_{12}\sin 2\theta,$$

$$A_{12} = \gamma_- \sin 2\theta + \gamma_{12}\cos^2\theta - \gamma_{21}\sin^2\theta,$$

$$A_{21} = \gamma_- \sin 2\theta + \gamma_{21}\cos^2\theta - \gamma_{12}\sin^2\theta, \quad (\text{A5})$$

where $\bar{\gamma}_{12} = (\gamma_{12} + \gamma_{21})/2$, and $\gamma_- = (\gamma_{11} - \gamma_{22})/2$. Now, for the considered symmetric damping matrix ($\gamma_{11} = \gamma_{22}$ and $\gamma_{12} = \gamma_{21}$), by taking $\theta = \pi/4$, the Lindblad operators in Eq. (A1) become diagonalized. Thus, the initially lossy system with incoherent mode coupling becomes a lossy system with coherent mode coupling with an NHH given in Eq. (30).

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