Generating high-order quantum exceptional points

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Recently, there has been intense research in proposing and developing various methods for constructing high-order exceptional points (EPs) in dissipative systems. These EPs can possess a number of intriguing properties related to, e.g., chiral transport and enhanced sensitivity. Proposals to realize high-order EPs have been based on the use of non-Hermitian Hamiltonians (NHHs) of composite systems, i.e., the operators describing the evolution of coupled post-selected systems or coupled intense light fields. In both cases, quantum jumps play no role. Here, by considering the full quantum dynamics of a quadratic Liouvillian superoperator, we introduce a simple and effective method for engineering NHHs with high-order quantum EPs, derived from evolution matrices of system operators moments. That is, by quantizing higher-order moments of system operators, e.g., of a quadratic two-mode system, the resulting evolution matrices can be interpreted as the new NHHs describing, e.g., networks of coupled resonators. Notably, such a mapping allows to correctly reproduce the results of the Liouvillian dynamics, including quantum jumps. By applying this mapping, we demonstrate that quantum EPs of any order can be engineered in dissipative systems and can, thus, be probed by the coherence and spectral functions. As an example, we consider a $U(1)$-symmetric quadratic Liouvillian describing an optical cavity with incoherent mode coupling, which can also possess anti-$\mathcal{PT}$-symmetry. Compared to their $\mathcal{PT}$-symmetric counterparts, such anti-$\mathcal{PT}$-symmetric systems could be easier to scale and, thus, can serve as a promising platform for engineering quantum systems with high-order EPs.

I. INTRODUCTION

Recently, the field of open quantum systems has attracted much interest. While dissipation is often seen as detrimental, there exist a whole class of processes which can never take place for Hermitian (i.e., non-dissipative) systems. In these systems, the existence of exotic spectral degenaracies called exceptional points (EPs) has attracted much attention [1–3]. At an EP, two or more eigenvalues, along with their eigenstates, coalesce. Since the eigenstates of a Hermitian operator are always orthogonal, EPs require non-Hermitian operators. Historically, EPs where first investigated in the context of non-Hermitian Hamiltonians (NHHs), primarily in the framework of parity-time ($\mathcal{PT}$)-symmetric systems, i.e., those for which a NHH commutes with the $\mathcal{PT}$ operator [4]. Note that non-Hermitian Hamiltonians do not lead to the violation of no-go theorems as explicitly demonstrated in Ref. [5]. The existence of EPs has been further generalized to any NHH exhibiting pseudo-Hermiticity [6], for which the $\mathcal{PT}$-symmetry is a particular case.

Beyond linear optical systems (see Refs. [1, 2] and references therein), EPs have been realized in various experimental platforms, e.g., in nonlinear optics [7–9], electronics [10], optomechanics [11–14], acoustics [15, 16], plasmonics [17], metamaterials [18], and ion trapped systems [19].

Many interesting and nontrivial effects are associated with the presence of EPs [20–34]. One of these is enhanced system sensitivity to external perturbations in the vicinity of EPs [35–44]. If n eigenstates coalesce (so the order of an EP is n), the response of a system to a perturbation of intensity ϵ scales as $\sqrt[n]{\epsilon}$. Although some recent studies (both theoretical and experimental) have questioned the presence of enhanced sensing at EPs [45–50], Refs. [48, 51, 52] have argued that EPs lead to enhanced sensitivity.

The interesting properties of high-order EPs ignited the search for methods which enable one to construct higher-order EPs [53–56]. For NHHs, the proposed techniques require to realize complex networks of coherently coupled resonators. A major drawback is that one has to finely tune the system parameters, due to the incommensurate mode couplings arising from the form of the mode coupling in NHHs [53–55]. In a recent work [56], the authors proposed a novel approach for constructing tight-binding networks with higher-order EPs, based on chiral-mode coupling instead.

Non-Hermiticity naturally emerges in the context of open quantum systems. Indeed, the Lindblad master equation of a Markovian open system, although Hermiticity preserving, has a well-defined arrow of time. Therefore, the Liouvillian superoperator associated with the

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master equation is non-Hermitian. With respect to an NHH, the Liouvillian also accounts for the presence of quantum jumps. The extension of EPs of NHHs to those based on Liouvillians [57] has shown that quantum jumps can play a crucial role in the properties of EPs [44, 58–63]. Furthermore, the evolution of a density matrix of an open quantum system is described by a completely-positive and trace-preserving (CPTP) linear map. As such, a NHH cannot describe the evolution of an arbitrary quantum system.

Despite the fact that an NHH may not describe the time evolution of a Lindblad master equation (i.e., the eigenstates of a NHH do not reproduce those of the Liouvillian), the dynamics of some operators can be derived in terms of the action of an NHH [63]. This apparent contradiction results from the fact that the operators in the Heisenberg picture do not evolve under a CPTP map. In other words, the NHH can describe the evolution of operators even in the quantum limit.

In this article, we propose a method to properly define a new class of effective NHHs for quadratic Liouvillian systems. These NHHs are derived from evolution matrices governing *the moments of system operators* and as such are called *moments-based NHHs*. Most importantly, these moments-based NHHs can reveal higher-order *quantum* EPs, residing in the Liouvillian eigenspace [63]. That is, by quantizing system operators moments, i.e., by mapping the corresponding evolution matrices to moments-based NHHs, one can engineer quantum systems with high-order EPs.

Such constructed moments-based NHHs with high-order EPs substantially differ from the *standard* NHHs. Whereas the latter are constructed by expanding the *Hilbert space* of system operators [53–55], the former are constructed by expanding rather the system operators *moments space*. Indeed, if one uses the eigenstates of the standard NHH for low-order operators moments to express higher-order ones, the result would be unphysical and give different results when compared to the full Liouvillian dynamics [63]. The moments-based NHHs, instead, are derived via the quantization of the system operators moments obtained from the Liouvillian, and, thus, correctly captures the dynamics of high-order system operators without approximations (i.e., including the effects of quantum jumps [57, 59]). The dynamics of such moments-based NHH can describe, e.g., a network of coupled resonators. In other words, by starting from a quadratic Liouvillian, describing, e.g., a two-mode system, the resulting evolution matrices, governing the higher-order field moments, can be cast to the new NHHs, which can describe networks of coupled cavities instead.

These newly obtained moments-based NHHs can reveal the presence of arbitrarily-high order quantum EPs in the Liouvillian dynamics. Physically speaking, one can witness the presence of high-order quantum EPs by means of the coherence and spectral functions [63] or by properly initializing the system. To put it another way, instead of considering *networks* of n resonators (where high-order EPs can be engineered), by considering higher-order moments of, e.g., *two coupled* resonators, one can obtain the *same* spectral degeneracies. Apart from a theoretical interest in defining the NHHs in the quantum limit, the advantage in the use of moments-based NHHs with respect to standard NHHs lies in its manifesting simplicity and in the possibility to preserve the commensurate character of modes coupling.

As an example, we implement our method for the $U(1)$-symmetric quadratic Liouvillian that describes a two-mode optical cavity with *incoherent* mode coupling. This model is also characterized by the anti-$\mathcal{PT}$-symmetry [63–68], as defined in Eq. (20). We show how the Liouvillian eigenspace of such a *two-mode* system, expressed via field moments, can be mapped to the eigenspace of an effective NHH of a *multimode* system. The benefit of considering anti-$\mathcal{PT}$-symmetric systems, compared to their $\mathcal{PT}$-symmetric counterparts with exclusively *coherent* mode coupling, is the absence of any active elements, and their scalability, which is seemingly easier to realize. That is, one does not need to build up complex networks to achieve higher-order EPs, but only *excite additional modes* in the anti-$\mathcal{PT}$-symmetric cavity; at the same time ensuring the incoherent character of the coupling between the newly excited modes. Recent studies show that such systems, with incoherent mode interactions, can be realized via incoherent mode backscattering in waveguide networks [69] and cavity-based photonic devices [70–73]. Moreover, the dissipative couplings can play a prominent role in the experimental realizations of photonic and quantum computing in time-multiplexed optical systems [74, 75].

As a byproduct of our method, we also highlight the rich structure of quadratic Liouvillians. Indeed, the correspondence between multimode systems and higher-order moments is a peculiarity of the Liouvillian space structure [44]. We argue that, although the correspondence between the Liouvillian evolution of higher-order correlation functions and lower-order correlation functions of more complex multimode systems is exact only for quadratic Liouvillians (i.e., describing Gaussian states), such a procedure should be valid also in the presence of weak nonlinearity, where a Gaussian state approximation can still be valid.

We note that our method can be implemented irrespective of the knowledge of the specific details of a given quadratic Liouvillian. That is, in order to realize a physical system with high-order EPs, initially one has only to have some physically realizable NHH (which of course can again be related to some quadratic Liouvillian), whose matrix mode representation reveals an EP at least of order two. Then, by increasing the order of the EP, according to the method described here, one obtains a non-Hermitian model, which can be a guide to realize extended lattice systems whose NHH has a higher-order EP.

The paper is organized as follows. In Sec. II, we introduce a general model of quadratic Liouvillians. In

Sec. III, we analyze the dynamics of higher-order moments of system operators in the model, expressed via the corresponding evolution matrices. Then, by performing a second quantization of the operator moments, we introduce a map between moments evolution matrices and the new class of NHHs, called moments-based NHHs, which can genuinely capture the quantum effects in a system. In Sec. IV, we describe a method to engineer higher-order EPs determined by the moments-based NHHs. In Sec. V, we implement the proposed scheme on the example of the $U(1)$ and anti-$\mathcal{PT}$-symmetric cavity with incoherent mode coupling. The discussion of the proposed method and its comparison to existing methods along with conclusions are given in Sec. VI.

II. GENERAL MODEL OF A SYSTEM DESCRIBED BY A QUADRATIC LIOUVILLIAN

The evolution of a density matrix $\hat{\rho}(t)$ is described by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = \mathcal{L}\hat{\rho}(t), \tag{1}$$

which for a quadratic Liouvillian superoperator $\mathcal{L}$ in the Gorini-Kossakowski-Sudarshan-Lindblad form reads ($\hbar = 1$)

$$\mathcal{L}\hat{\rho}(t) = -i\left(\hat{H}_{\text{eff}}\hat{\rho}(t) - \hat{\rho}(t)\hat{H}_{\text{eff}}^{\dagger}\right) + 2\sum_{jkl}\Gamma_{jk}^{l}\hat{s}_j^{(l)}\hat{\rho}(t)\left(\hat{s}_k^{(l)}\right)^{\dagger}, \tag{2}$$

where $\hat{H}_{\text{eff}}$ is the effective NHH given by:

$$\hat{H}_{\text{eff}} = \sum \kappa_{jk}^{lm}\hat{s}_j^{l}\hat{s}_k^{m} - i\sum\Gamma_{jk}^{l}\hat{s}_j^{(l)}\left(\hat{s}_k^{(l)}\right)^{\dagger}. \tag{3}$$

In Eqs. (2) and (3), the indices $\{j,\, k\}$ indicate the sites (modes) of a system, and $\{l,\, m\} = \{1,\, 2\}$ are such that $\hat{s}_q^{(1,2)} = \{\hat{a}_q, \hat{a}_q^{\dagger}\}$, where $\hat{a}_q$ ($\hat{a}_q^{\dagger}$) is the annihilation (creation) operator of a particle at the qth site (e.g., a photon). The coefficients $\kappa_{jk}^{lm}, \Gamma_{jk}^{lm} \in \mathbb{R}$ describe the coherent and incoherent parts of the system evolution, respectively. Such a quadratic Liouvillian describes the dissipative dynamics of Gaussian states that, in optics, describes, e.g., linearly coupled waveguides (cavities) or nonlinear parametric processes [76].

From now on, we will describe $\hat{H}_{\text{eff}}$ in Eq. (3) as the *effective NHH* to distinguish it from the *moments-based NHH*, which is associated with system operators moments of higher order and which we will introduce in Sec. III C.

According to the quantum trajectory theory, the Liouvillian in Eq. (2) can be divided in two parts; namely, in a *continuous* nonunitary evolution described by the effective NHH $\hat{H}_{\text{eff}}$, and in the action of *discrete* random changes expressed by quantum jumps [77, 78]. The effective NHH is especially useful in the semiclassical limit, where the jump action can be neglected. It can also be used to describe systems where it is possible to determine if a quantum jump took place, e.g., cavities with a small photon number and with a very high finesse [79] or postselected systems [80]. In other cases, the last term in Eq. (2), describing the quantum jumps effects (in this case, a sudden creation or annihilation of a particle) is essential to faithfully capture the system dynamics at the quantum level. Indeed, for the case studied here of open *linear* systems, the effective NHH $\hat{H}_{\text{eff}}$ does not reflect the non-conservative character of dissipation [60].

III. DYNAMICS OF THE MOMENTS OF SYSTEM OPERATORS AND THE NON-HERMITIAN HAMILTONIAN

In the following, we assume that the dissipators in Eq. (2) induce no incoherent amplification. Otherwise, the dynamics of some moments is affected by an additional noise vector [81, 82], which is of no relevance here.

A. Field moments evolution for a quadratic $U(1)$-symmetric two-mode Liouvillian

We begin by considering the simplest case, i.e., that of a quadratic and $U(1)$ symmetric systems describing two cavities (its generalization is provided in Sec. III B). We refer to the corresponding $\mathcal{L}$ as a *linear* Liouvillian because it emerges in the context of dissipative coupled linear systems, e.g., coupled waveguides or cavities.

Any Liouvillian for coupled bosonic systems is said to be $U(1)$-symmetric if it commutes with a phase rotation operator $\mathcal{U}$, defined as

$$\mathcal{U}\hat{\rho} = \exp\left(i\phi\sum_j \hat{a}_j^{\dagger}\hat{a}_j\right)\hat{\rho}\exp\left(-i\phi\sum_j \hat{a}_j^{\dagger}\hat{a}_j\right). \tag{4}$$

In other words, the master equation must be invariant under a simultaneous arbitrary phase shift $\hat{a}_j \to \hat{a}_j e^{i\phi}$.

For two coupled cavities, i.e., $j,\, k = 1,\, 2$ in Eq. (2), $\mathcal{U}$ constraints the rate equations for the *field moments*

$$\langle \hat{a}_1^{\dagger m}\hat{a}_2^{\dagger n}\hat{a}_1^{p}\hat{a}_2^{q}\rangle, \quad \text{for} \quad m,n,p,q = 0,1,\ldots.$$

Normally, the dynamics of the field moments relates all the possible combinations of $\{m,n,p,q\}$. However, in the presence of the $\mathcal{U}$ symmetry only moments that have the same order ($m+n-p-q$) can be coupled. Since the considered Liouvillians here are also quadratic, one obtains $p+q = m+n$. Thus, the dynamics of the moments is captured by *closed sets of coupled equations*. For example, the first-order moment $\langle\hat{a}_1\rangle$ ($\langle\hat{a}_1^{\dagger}\rangle$) would be coupled only to the moment $\langle\hat{a}_2\rangle$ ($\langle\hat{a}_2^{\dagger}\rangle$).

Given the closed structure of the rate equations for the first-order moments we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{A}\rangle = \boldsymbol{M}_A\langle\vec{A}\rangle, \tag{5}$$

where $\vec{A} = [\hat{a}_1, \hat{a}_2]^T$, and $\boldsymbol{M}_A$ is a 2×2 evolution matrix.

The matrix $\boldsymbol{M}_A$ is the building block to obtain the evolution matrix for higher-order field moments by constructing various Kronecker products of the vectors $\vec{A}$ and $\vec{A}^\dagger \equiv [\hat{a}_1^\dagger, \hat{a}_2^\dagger]$. For instance, given the second-order moments

$$\langle \vec{B}\rangle = \langle \vec{A}\otimes\vec{A}\rangle = \langle [\hat{a}_1^2, \hat{a}_2\hat{a}_1, \hat{a}_1\hat{a}_2, \hat{a}_2^2]^T\rangle, \tag{6}$$

the evolution matrix $\boldsymbol{M}_B$, such that $\partial_t\langle\vec{B}\rangle = \boldsymbol{M}_B\langle\vec{B}\rangle$, is the Kronecker sum of the same two matrices $\boldsymbol{M}_A$,

$$\boldsymbol{M}_B = \boldsymbol{M}_A \oplus \boldsymbol{M}_A = \boldsymbol{M}_A\otimes \boldsymbol{I}_2 + \boldsymbol{I}_2\otimes\boldsymbol{M}_A. \tag{7}$$

Here, the symbol $\oplus$ denotes a Kronecker sum, $\otimes$ is the Kronecker product, and $\boldsymbol{I}_2$ is the 2×2 identity matrix. Note that we keep the order between the products of the operators $\hat{a}_1$ and $\hat{a}_2$ in Eq. (6). Although, in this case, it is not relevant because $[\hat{a}_1, \hat{a}_2] = 0$, but in general the order should be preserved. We remark that a Kronecker sum naturally appears in problems when solving Lyapunov and/or Sylvester equations [83].

The same procedure can recursively be applied to obtain the evolution matrices for higher-order moments. The generalization of Eq. (7) is given by:

$$\langle\vec{\gamma}\rangle = \langle\vec{\alpha}\otimes\vec{\beta}\rangle,$$
$$\boldsymbol{M}_\gamma = \boldsymbol{M}_\alpha\oplus\boldsymbol{M}_\beta = \boldsymbol{M}_\alpha\otimes\boldsymbol{I}_\beta + \boldsymbol{I}_\alpha\otimes\boldsymbol{M}_\beta, \tag{8}$$

where the vectors of operators, $\vec{\alpha}$ and $\vec{\beta}$, are the Kronecker products of the initial vectors $\vec{A}$ and/or $\vec{A}^\dagger$, while $\boldsymbol{M}_{\alpha,\beta}$ are the corresponding evolution matrices. The resulting evolution matrix $\boldsymbol{M}_\gamma$ of higher-order field moments is the recursive Kronecker sum of the matrices $\boldsymbol{M}_A$. The dimension N_γ of the matrix $\boldsymbol{M}_\gamma$ is the product of the dimensions of the matrices $\boldsymbol{M}_\alpha$ and $\boldsymbol{M}_\beta$, i.e., $N_\gamma = N_\alpha N_\beta$. Moreover, due to the standard properties of the Kronecker sum, the symmetry of the matrix $\boldsymbol{M}_A$ is retained by all the matrices $\boldsymbol{M}_\gamma$.

B. Evolution of field moments for a generic quadratic Liouvillian

The previous derivation for a $U(1)$ quadratic two-mode system can be extended to generic quadratic Liouvillians, even if the form of $\boldsymbol{M}_\gamma$ is slightly more involved.

For an arbitrary quadratic Liouvillian (i.e., not necessarily $U(1)$-symmetric), describing an n-mode system, all field moments are generated by the tensor product of the $2n$-dimensional vector

$$\vec{A} = [\hat{a}_1, \hat{a}_1^\dagger, \dots, \hat{a}_n, \hat{a}_n^\dagger]^T. \tag{9}$$

The time evolution of $\langle\vec{A}\rangle$ is given by $\partial_t\langle\vec{A}\rangle = \boldsymbol{M}_A\langle\vec{A}\rangle$, which is the same as in Eq. (5).

Given the building block $\boldsymbol{M}_A$, the previously detailed procedure, to obtain the evolution matrices for higher-order moments, remains valid. Similarly to the $U(1)$ case, the evolution matrices $\boldsymbol{M}_\gamma$, which determine the dynamics of various higher-order moments, are obtained by taking a recursive Kronecker sum of the corresponding $2n\times 2n$ matrix $\boldsymbol{M}_A$. That is, Eq. (8) is valid for any quadratic Liouvillian. Also, as it was stressed earlier, the operators order should be kept when constructing Kronecker tensors out of the vector of operators $\vec{A}$ in Eq. (9). In other words, no permutations are allowed in the obtained products of the operators.

C. Second quantization of field moments and a definition of the moments-based non-Hermitian Hamiltonian

1. *Moments-based non-Hermitian Hamiltonian of $U(1)$ quadratic Liouvillians*

In the case of $U(1)$ quadratic Liouvillians, the evolution matrix $\boldsymbol{M}_A$ for the first-order field moments becomes equivalent (up to the imaginary factor) to the matrix form of the corresponding NHH, i.e.,:

$$\boldsymbol{M}_A = -i\boldsymbol{H}_{\text{eff}}, \tag{10}$$

where the matrix form of the NHH is defined as follows

$$\hat{H}_{\text{eff}} \equiv \left(\vec{A}\right)^\dagger \boldsymbol{H}_{\text{eff}}\vec{A}, \tag{11}$$

and vector of operators $\vec{A}$, for a two-mode case, is given in Eq. (5). In other words, there is a one-to-one correspondence between the evolution of the operator $\vec{A}$ and its first-order moments $\langle\vec{A}\rangle$ for the $U(1)$ systems. This correspondence is quite intriguing, since it provides a clear physical meaning to the effective NHH of a two coupled bosonic systems via the introduction of its first-order moments, and which has been intensively exploited in a number of previous works on *quantum* EPs [59, 84, 85].

The described above correspondence however cannot be simply extended to higher-order field moments, since the evolution matrices governing the dynamics of the operators and their moments would be in general different. This stems from the fact that the dynamics of the field moments of any order is determined by the Liouvillian which include quantum jump effects, whereas the effective NHH applied to the same moments (in Heisenberg picture) in general fails to incorporate them [63].

Nevertheless, one can assign to any evolution matrix $\boldsymbol{M}_\gamma$ a new NHH in analogy to Eq. (10), which we thus call a moments-based NHH, by quantizing the corresponding higher-order field moments $\langle\vec{\gamma}\rangle$ (see also Fig. 1). The determination of such moments-based NHHs however requires an intermediate passage. Namely, in general, $\vec{\gamma}$ contains terms which are identical [e.g., $\hat{a}_1\hat{a}_2 = \hat{a}_2\hat{a}_1$ in Eq. (6)], and it cannot be straightforwardly quantized. The degeneracy of $\boldsymbol{M}_\gamma$ can be eliminated by introducing the *reduced* matrix $\boldsymbol{M}_\gamma^{\text{red}}$. For instance, $\boldsymbol{M}_\gamma$ of any non-Hermitian moment of the vector $\vec{A}$, i.e., $\vec{\gamma} = \bigotimes_{i=1}^{m}\vec{A}$, has an "initial" dimension $N_\gamma = 2^m$. By collecting identical terms, the resulting matrix $\boldsymbol{M}_\gamma^{\rm red}$ has the dimension $N_\gamma^{\rm eff} = m+1$. The price to pay for such reduction is, in general, the loss of some initial symmetry of the matrix $\boldsymbol{M}_\gamma$. For instance, if $\boldsymbol{M}_\gamma = \boldsymbol{M}_\gamma^T$ then, in general, $\boldsymbol{M}_\gamma^{\rm red} \neq \left(\boldsymbol{M}_\gamma^{\rm red}\right)^T$. That is, if the mode coupling in the matrix $\boldsymbol{M}_\gamma$ is symmetric, then that coupling in the effective evolution matrix $\boldsymbol{M}_\gamma^{\rm eff}$ is, in general, asymmetric.

Having eliminated the redundant variables, any N dimensional vector $\hat\gamma$ in Eq. (8) can be quantized as $\vec\gamma \to \vec\gamma'$, where $\vec\gamma' = [\hat b_1, \ldots, \hat b_N]$ is the vector of the boson annihilation operators $\hat b_j$, which describe new fields, as shown in Fig. 1. The emerging physics is that a *new linear dissipative system* is constructed. This new system is described by N coupled fields $\hat b_j$ and it evolves under a moments-based NHH $\hat H_\gamma^{\rm mb}$ given by [c.f. Eq. (10)]:

$$\boldsymbol{H}_\gamma^{\rm mb} = i\boldsymbol{M}_\gamma^{\rm red}, \tag{12}$$

where $\boldsymbol{H}_\gamma^{\rm mb}$ is, as before, a matrix form of the moments-based NHH $\hat H_\gamma^{\rm mb} = (\vec\gamma')^\dagger \boldsymbol{H}_\gamma^{\rm mb}\vec\gamma'$.

This procedure is easily and recursively extended to any set of higher-order moments. Thus, moments-based NHHs representing large systems can be constructed by considering an initial $U(1)$ two-mode system with an evolution matrix $\boldsymbol{M}_A$ for the first-order field moments. Vice versa, one can obtain higher-order EPs by using the moments-based NHHs as a guideline to realize dissipative lattices of coupled resonator, whose effective NHH will have higher-order EPs (see Fig. 1). Notably, the effective NHH of the lattice could be characterized by a smaller decay rate of the observables, resulting in a better visibility of the EP.

2. *Moments-based non-Hermitian Hamiltonian of generic quadratic Liouvillians*

In a general case, the correspondence between $\boldsymbol{M}_A$ and the matrix form of the NHH $\hat H_{\rm eff}$, drawn in Eq. (10), does not hold anymore. Instead, one has:

$$\boldsymbol{H}_{\rm eff} \equiv i\boldsymbol{\eta}_1\boldsymbol{M}_A + i\boldsymbol{\eta}_2\boldsymbol{M}_A^\dagger\boldsymbol{\eta}_3, \tag{13}$$

where

$$\boldsymbol{\eta}_1 = \widetilde{\bigoplus_n}\begin{pmatrix}1 & \\ & 0\end{pmatrix},\ \boldsymbol{\eta}_2 = \widetilde{\bigoplus_n}\begin{pmatrix}0 & \\ & 1\end{pmatrix},\ \boldsymbol{\eta}_3 = \widetilde{\bigoplus_n}\begin{pmatrix}-1 & \\ & 1\end{pmatrix}. \tag{14}$$

In Eq. (14), the symbol $\widetilde{\bigoplus}$ means a direct sum (not the Kronecker one). In Eq. (13), without loss of generality, we have also dropped a constant term related to the field frequencies.

Equation (13) reads as follows. The first term in its r.h.s. takes into account the dynamics of only the annihilation operators $\hat a_k$, i.e., the odd elements of the vector $\hat A$ in Eq. (9). This term is equivalent to the r.h.s. of Eq. (10), when the system is $U(1)$ symmetric and linear. The second term in Eq. (13) accounts for the dynamics of the creation operators of the fields, i.e., the even elements of the vector $\hat A$.

Contrary to the case of the linear systems, Eq. (13) implies that the evolution matrix $\boldsymbol{M}_A$ and the NHH $\hat H_{\rm eff}$ are, in general, not simply related. Even though the critical points of the NHH and Eq. (13) coincide, the spectral properties of these matrices might differ. And it is the spectral degeneracies of the evolution matrices that determine the properties of the coherence and spectral functions, which are experimentally accessible via photocount or homodyne measurements.

Despite the fact that now there is no one-to-one correspondence between the evolution matrix for the first-order field moments and effective NHH in the case of generic nonlinear quadratic Liouvillians, according to Eq. (13), one still can map the evolution matrix for higher-order moments to a new moments-based NHH, i.e.,

$$\boldsymbol{M}_\gamma \to \boldsymbol{H}_\gamma^{\rm mb}. \tag{15}$$

The mapping in Eq. (15), in general, might require additional operations over the matrix $\boldsymbol{M}_\gamma$, compared to Eqs. (12) and (13), e.g., row and column permutations, which depends on how a corresponding vector of operators $\vec\gamma$ is constructed from the initial vector $\vec A$.

As it has been shown in Ref. [86], even for a class of *Hermitian* quadratic Hamiltonians, i.e., without dissipation, and which describes optical nonlinear processes, the corresponding evolution matrix $\boldsymbol{M}_A$ can reveal an EP. Moreover, one can easily map the matrix $\boldsymbol{M}_A$ to a new NHH as in Eq. (10) [86]. Based on the latter, one then can straightforwardly apply Eq. (12) for the construction of new moments-based NHHs exhibiting higher-order EPs.

Within this description, we can properly define the moments-based NHH of composite systems of higher-order moments. The advantages of this procedure are manifold: (i) The moments-based NHH now correctly takes into account the effects of quantum jumps. (ii) These NHH can now be correctly quantized, and the commutation rules emerging from the physical moments-based NHH can correspond to those of the original system. In other words, the computation of quantum-relevant variables is unaffected by the algebraic construction. (iii) The procedure can be easily implemented numerically, even for large systems.

IV. ENGINEERING HIGHER-ORDER EXCEPTIONAL POINTS FROM QUADRATIC LIOUVILLIANS

In the previous discussion, we proved that it is possible to define a physical NHH using the full-Liouvillian description if one focuses on the dynamics of moments.

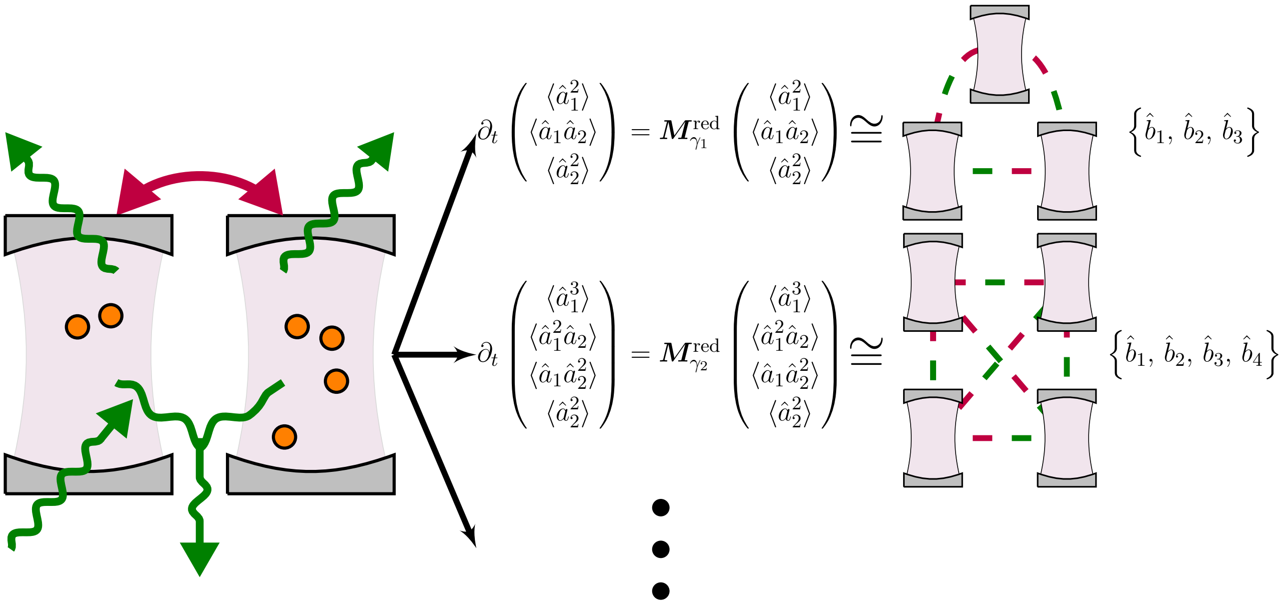

FIG. 1. Schematic representation of the procedure to define a moments-based NHH and its relation to a lattice system of resonators in the $U(1)$ symmetric case. A quadratic Liouvillian system (e.g., the two cavities on the left) is characterized by coherent interactions (the red double arrows), dissipation (green arrows pointing outwards), and amplification channels (green arrows pointing inwards) that compete in determining the photonic field inside the cavity (orange balls). The equations of motion for the various moments $\langle \hat a_1^{\dagger m} \hat a_2^{\dagger n} \hat a_1^p \hat a_2^q \rangle$ of such a system (middle part) form a finite set, which can be described by the reduced evolution matrices $\boldsymbol{M}_\gamma^{\rm red}$ (see the main text in Sec. III C 1 for details). By quantizing the moments, such matrices $\boldsymbol{M}_\gamma^{\rm red}$ can be interpreted as an effective NHH of a more complex lattice of resonators (right part of the figure), according to Eq. (12). Indeed, each moment can be mathematically treated as a separate bosonic field ($\hat b_i$), i.e., a driven dissipative quadratic bosonic system, as shown in the right panel with cavities. Each of this field interact with the others via dissipative or coherent interactions (represented by green and red dashed lines). This procedure is general, and explores a direct correspondence between the evolution of higher-order moments of a Liouvillian system and lower-order moments of a larger system. Interestingly, this correspondence can be exploited to observe higher-order EPs in simple lattices by considering higher-order correlation functions and, vice versa, as a guideline to engineer lattices which display higher-order EPs.

Despite the lack of the correspondence between the eigenvectors describing the state and operators evolution (namely, the right- and left-hand-side eigenstates of the Liouvillian) there is a correspondence between their eigenvalues and their degeneracies. Indeed, if the states have an EP, so do the moments. Here, by considering the spectra of evolution matrices for higher-order field moments, and exploiting the described above mapping, we show how to engineer moments-based NHHs with higher-order EPs in any quadratic Liouvillian systems.

A. High-order exceptional points

Due to the structure of the equations of motion of the moments, the spectral degeneracies of an evolution matrix $\boldsymbol{M}_\gamma$ can be directly obtained from those of $\boldsymbol{M}_A$. Given the Kronecker sum in Eq. (8), one has the following relation between eigenvalues of the matrices [83]:

$$\lambda_{ij}(\boldsymbol{M}_\gamma) = \lambda_i(\boldsymbol{M}_\alpha) + \lambda_j(\boldsymbol{M}_\beta), \tag{16}$$

with $i = 1, \ldots, N_\alpha$ and $j = 1, \ldots, N_\beta$. Therefore, with each new term $\boldsymbol{M}_A$ in Eq. (8), the order of an EP of $\boldsymbol{M}_\gamma$ increases by one. Note that, although the *algebraic* degeneracy of the eigenvalues in Eq. (16) grows proportionally to N_γ, its *geometric* multiplicity does not. Indeed, if $\boldsymbol{M}_A$ has an EP of order two, the evolution matrix $\boldsymbol{M}_\gamma = \bigoplus_{i=1}^{m} \boldsymbol{M}_A$ has an EP of order $(m+1)$.

This result is an analogous prove of that in Ref. [63], were it was demonstrated that once the evolution matrix $\boldsymbol{M}_A$ has an EP of second order, it immediately implies the presence of an EP of any higher-order $n \geq 2$ in the Liouvillian eigenspace. This eigenspace determines the evolution matrices $\boldsymbol{M}_\gamma$ of field moments. Consequently, the quantized moments and the corresponding NHHs given in Eqs. (12) and (15) are characterized by the same degeneracy as that of $\boldsymbol{M}_\gamma$.

The matrix $\boldsymbol{M}_\gamma$ determines also the dynamics of high-order correlation functions, according to the quantum regression theorem [82]. The Wick's theorem for Gaussian states (i.e., quadratic systems) indicates that correlation functions of any order can be expressed as a sum of products of correlation functions of lower orders. This implies that if the matrix $\boldsymbol{M}_A$, which determines the first-order coherence function, exhibits an EP, higher-order coher-

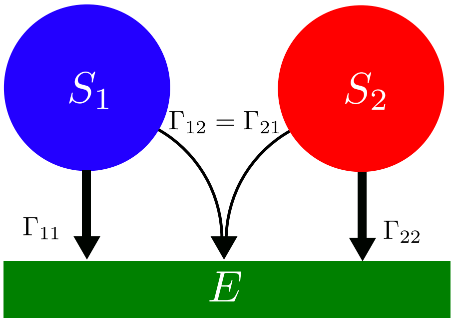

FIG. 2. Schematic representation of the model described by the Liouvillian superoperator $\mathcal{L}$ in Eq. (17). S_1 and S_2 are two system modes, which dissipate to the environment E with rates Γ_{11} and Γ_{22}, respectively. The modes are also dissipatively coupled to the environment with the incoherent coupling strength $\Gamma_{12} = \Gamma_{21}$.

ence functions reveal a higher order EP [63]. The same conclusion can alternatively be drawn using the properties of the matrix exponential of a Kronecker sum [87].

Nevertheless, when constructing the moments-based NHH, the reduced matrix $\boldsymbol{M}_\gamma^{\rm red}$ is used instead, according to Eqs. (12) and (15). As a result, the order of an EP of the reduced matrix would correspond to its dimension, e.g., for the $U(1)$ case, $N(\boldsymbol{M}_\gamma^{\rm red}) = m+1$, which is the same as the order of the EP. As a result, the corresponding moments-based NHH $\hat{H}_\gamma^{\rm mb}$, describing $(m+1)$ modes (resonators) would have an EP of the order $(m+1)$.

Moreover, from Eqs. (8) and (16) it is evident that the construction of the moments-based NHH $\hat{H}_\gamma^{\rm mb}$ for higher-order EPs can be realized with arbitrary matrices $\boldsymbol{M}_\alpha$ and $\boldsymbol{M}_\beta$. We also note that although our method implicitly assumes that the evolution matrix $\boldsymbol{M}_A$ for first-order field moments already has an EP, there are methods which allow to construct a new matrix having an EP from a combination of generic complex matrices with no initial degeneracies [88].

V. EXAMPLE OF A $U(1)$ ANTI-$\mathcal{PT}$-SYMMETRIC CAVITY

In this section, we implement our scheme for the example of a $U(1)$-symmetric two-mode cavity with incoherent mode coupling, which additionally possesses the anti-$\mathcal{PT}$-symmetry. Namely, we show how a Liouvillian eigenspace of such a two-mode system, expressed via field moments and their evolution matrices, can be mapped to a new moments-based NHH representing a multimode system.

In other words, first we reveal EPs of any order arising from the evolution matrices of the bimodal system under consideration. Then, for any evolution matrix which exhibits an EP of order $n > 2$, we assign a new moments-based NHH, which can, thus, correspond to a new coupled n-mode system with higher-order EPs.

A. Model of an anti-$\mathcal{PT}$-symmetric bimodal cavity

The model under consideration is the same as in Ref. [63]. Namely, we consider the Lindblad master equation in Eq. (1) with the following Liouvillian superoperator [89–91]:

$$\mathcal{L}\hat{\rho} = -i\left(\hat{H}_{\rm eff}\hat{\rho} - \hat{\rho}\hat{H}_{\rm eff}^\dagger\right) + \sum_{j,k=1,2} \Gamma_{jk}\hat{a}_j\hat{\rho}\hat{a}_k^\dagger, \tag{17}$$

with the effective NHH:

$$\hat{H}_{\rm eff} = \sum_{j=1,2} \omega_j \hat{a}_j^\dagger \hat{a}_j - i \sum_{j,k=1,2} \Gamma_{jk}\hat{a}_j^\dagger\hat{a}_k. \tag{18}$$

The Liouvillian in Eq. (17) describes a dissipative linear system of two incoherently coupled modes. A schematic diagram of the model under study is shown in Fig. 2. In Eq. (17), $\hat{a}_j$ ($\hat{a}_j^\dagger$) is the annihilation (creation) operator of mode j with a bare frequency ω_j; the diagonal damping coefficient Γ_{kk} denotes the inner kth mode decay rate, while the off-diagonal coefficient $\Gamma_{jk} = \Gamma_{kj}$ (for $j \neq k$) accounts for the *incoherent* coupling strength between modes j and k, due to the interaction of both modes with the environment [89]. That is, without loss of generality, we focus on a symmetric form of the decoherence matrix in Eq. (17), although, in general, $\Gamma_{jk} \neq \Gamma_{kj}$. The latter case can also result into the chiral character of the interaction between modes.

B. Second-order EP

In our previous study [63], we analyzed EPs, up to their third order, of such an anti-$\mathcal{PT}$-symmetric bimodal cavity. The evolution matrix for the first-order field moments $\langle\hat{A}\rangle = [\langle\hat{a}_1\rangle, \langle\hat{a}_2\rangle]^T$ in Eq. (5) takes the form [63]:

$$\boldsymbol{M}_A = \begin{pmatrix} -i\Delta - \Gamma & -\Gamma_{12} \\ -\Gamma_{12} & i\Delta - \Gamma \end{pmatrix}, \tag{19}$$

where $\Delta = \omega_2 - \omega_1$ is the frequency difference between the two modes, $\Gamma = \Gamma_{11} = \Gamma_{22}$ is an inner loss rate of each mode, and $\Gamma_{12} = \Gamma_{21}$ is an incoherent mode coupling strength (for details, see Ref. [63]).

According to Eq. (19), the corresponding NHH is anti-$\mathcal{PT}$-symmetric, since it anticommutes with the parity-time $\mathcal{PT}$ operator, i.e.,

$$\mathcal{PT}\hat{H}_{\rm eff}\mathcal{PT} = -\hat{H}_{\rm eff}, \tag{20}$$

which implies the $\mathcal{PT}$-symmetry of the evolution matrix $\boldsymbol{M}_A$. Moreover, by appropriately rotating the anti-$\mathcal{PT}$-symmetric NHH $\hat{H}_{\rm eff}$, one can switch it to a passive $\mathcal{PT}$-symmetric system [63].

The eigenvalues of the matrix $\boldsymbol{M}_A$ in Eq. (19) read

$$\lambda_{1,2} = -\Gamma \pm \sqrt{\Gamma_{12}^2 - \Delta^2}. \tag{21}$$

Thus, the EP of the system (which is of second order for the evolution matrix $\boldsymbol{M}_A$) is observed at the point

$$\Gamma_{12}^{\text{EP}} = |\Delta|. \tag{22}$$

As a consequence, at the EP, the matrix $\boldsymbol{M}_A$, and thus $\hat{H}_{\text{eff}}$, acquire a Jordan form, i.e., it become non-diagonalizable.

C. Third-order EP

According to Eqs. (6) and (7), the matrix $\boldsymbol{M}_B = \boldsymbol{M}_A \oplus \boldsymbol{M}_A$ written for the non-Hermitian second-order moments $\langle\vec{B}\rangle$ [in the form given in Eq. (6)] reads as:

$$\boldsymbol{M}_B = \begin{pmatrix} -2i\Delta - 2\Gamma & -\Gamma_{12} & -\Gamma_{12} & 0 \\ -\Gamma_{12} & -2\Gamma & 0 & -\Gamma_{12} \\ -\Gamma_{12} & 0 & -2\Gamma & -\Gamma_{12} \\ 0 & -\Gamma_{12} & -\Gamma_{12} & 2i\Delta - 2\Gamma \end{pmatrix}, \tag{23}$$

would attain the same EP but of its third order. Indeed, the eigenvalues of this matrix are:

$$\lambda_{1,2} = -2\Gamma \pm 2\sqrt{\Gamma_{12}^2 - \Delta^2}, \quad \lambda_{3,4} = -2\Gamma, \tag{24}$$

which at the EP in Eq. (22) become identical. The plots for these eigenvalues were presented in Ref. [63]. Although the *algebraic* multiplicity of the eigenvalues in Eq. (24) at the EP equals four, the *geometric* multiplicity equals three; thus, indicating the coalescence of three modes. The latter fact points that the EP is, indeed, of third order. Moreover, for the matrix $\boldsymbol{M}_B$ in Eq. (23), as mentioned above, one can effectively reduce its dimension to three, keeping the same order of the EP. That is, by reducing the vector of moments

$$\begin{pmatrix} \langle\hat{a}_1^2\rangle \\ \langle\hat{a}_2\hat{a}_1\rangle \\ \langle\hat{a}_1\hat{a}_2\rangle \\ \langle\hat{a}_2^2\rangle \end{pmatrix} \rightarrow \begin{pmatrix} \langle\hat{a}_1^2\rangle \\ \langle\hat{a}_1\hat{a}_2\rangle \\ \langle\hat{a}_2^2\rangle \end{pmatrix}, \tag{25}$$

one obtains the following effective matrix

$$\boldsymbol{M}_B \rightarrow \boldsymbol{M}_B^{\text{red}} = \begin{pmatrix} -2i\Delta - 2\Gamma & -2\Gamma_{12} & 0 \\ -\Gamma_{12} & -2\Gamma & -\Gamma_{12} \\ 0 & -2\Gamma_{12} & 2i\Delta - 2\Gamma \end{pmatrix}. \tag{26}$$

By comparing Eqs. (23) and (26), one can see that the reduced matrix $\boldsymbol{M}_B^{\text{red}}$ has lost the symmetry of the initial matrix $\boldsymbol{M}_B$, i.e., $\boldsymbol{M}_B^{\text{red}} \neq (\boldsymbol{M}_B^{\text{red}})^T$.

One can directly obtain a NHH from $\boldsymbol{M}_B^{\text{red}}$, according to Eq. (12). Namely, by additionally discarding the inner decoherence terms 2Γ in Eq. (26), the corresponding dissipative system for this model is a lattice of three bosonic modes, interacting via the effective NHH

$$\hat{H}_{\text{eff}} = 2\Delta\left(\hat{b}_1^\dagger\hat{b}_1 - \hat{b}_3^\dagger\hat{b}_3\right) - i\Gamma_{12}\left(\hat{b}_1\hat{b}_2^\dagger + 2\hat{b}_1^\dagger\hat{b}_2 + \hat{b}_3\hat{b}_2^\dagger + 2\hat{b}_3^\dagger\hat{b}_2\right), \tag{27}$$

and whose explicit Liouvillian dynamics reads

$$\mathcal{L}\hat{\rho}(t) = -i(\hat{H}_{\text{eff}}\hat{\rho}(t) - \hat{\rho}(t)\hat{H}_{\text{eff}}) + 2\sum_{i,j}\gamma_{ij}\left(\hat{b}_i\hat{\rho}\hat{b}_j^\dagger\right), \tag{28}$$

where the decoherence matrix in Eq. (28) has only nonzero off-diagonal elements $\gamma_{ji} = 2\gamma_{ij} = 2\Gamma_{12}$ with $i = 1, 3,\ j = 2$.

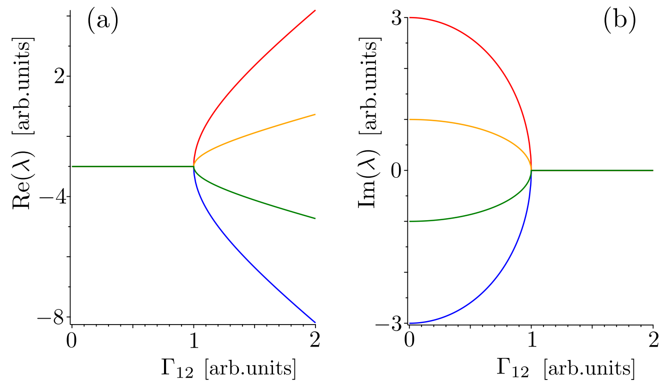

FIG. 3. (a) Real and (b) imaginary parts of the eigenvalues λ, according to Eq. (31), of the effective evolution matrix $\hat{M}_C^{\text{eff}}$, given in Eq. (30), for the third-order field moments. Its four eigenvalues coalesce at the EP in Eq. (22), thus indicating that the EP is of the fourth order. System parameters are set as $\Gamma = 1$ [arb. units] and $\Delta = 1$ [arb. units].

D. Fourth-order EP

In a similar way, one can generate an NHH with an EP of the fourth order, whose matrix form reads as follows

$$\boldsymbol{H}_\gamma^{\text{mb}} = i\boldsymbol{M}_C = i\bigoplus_{n=1}^{3}\boldsymbol{M}_A, \tag{29}$$

where $\vec{\gamma} = \vec{A} \otimes \vec{A} \otimes \vec{A}$, and $\boldsymbol{M}_A$ is given in Eqs. (23) and (19), respectively. Note that the resulting matrix $\hat{M}_C$ has dimension $2^3 \times 2^3$. In other words, the number of required modes increases to eight. Nonetheless, again, one can contract the resulting matrix to dimension four, but at the expense of losing the mode-coupling symmetry. Namely, the evolution matrix $\boldsymbol{M}_C$ can be defined by eight moments of the form $\langle\hat{a}_i\hat{a}_j\hat{a}_k\rangle$, where $i, j, k = 1, 2$. As such, there are two sets of three equivalent moments $\langle\hat{a}_i^2\hat{a}_j\rangle = \langle\hat{a}_i\hat{a}_j\hat{a}_i\rangle = \langle\hat{a}_j\hat{a}_i^2\rangle$ for $i, j = 1, 2$ and $i \neq j$. As a result, only *four* non-degenerate moments remain out of *eight*, which, thus, defines the 4×4 reduced matrix

$\boldsymbol{M}_C^{\rm red}$, and which attains the following form

$$\boldsymbol{M}_C^{\rm red} = \begin{pmatrix} -3i\Delta-3\Gamma & -3\Gamma_{12} & 0 & 0 \\ -\Gamma_{12} & -i\Delta-3\Gamma & -2\Gamma_{12} & 0 \\ 0 & -2\Gamma_{12} & i\Delta-3\Gamma & -\Gamma_{12} \\ 0 & 0 & -3\Gamma_{12} & 3i\Delta-3\Gamma \end{pmatrix}, \tag{30}$$

and its eigenvalues read

$$\lambda_{1,2} = -3\Gamma \pm \sqrt{\Gamma_{12}^2-\Delta^2},$$
$$\lambda_{3,4} = -3\Gamma \pm 3\sqrt{\Gamma_{12}^2-\Delta^2}. \tag{31}$$

We plot these eigenvalues in Fig. 3.

E. $(N+1)$th-order EP

Clearly, one can obtain the effective evolution matrices for any dimension $(N+1)$ as

$$\boldsymbol{M}_{N+1}^{\rm eff} = \begin{pmatrix} \Delta_0 & -N\Gamma_{12} & & & \\ \dots & \dots & \dots & \dots & \dots \\ & -n\Gamma_{12} & \Delta_n & -(N-n)\Gamma_{12} & \\ \dots & \dots & \dots & \dots & \dots \\ & & & -N\Gamma_{12} & \Delta_N \end{pmatrix}, \tag{32}$$

where $\Delta_n = i(-N+2n)\Delta - N\Gamma$ for $n=0,\dots,N$. Combining now Eqs. (16) and (21), the eigenvalues of this matrix can be written as

$$\lambda_n = -N\Gamma \pm (N-2n)\sqrt{\Gamma_{12}^2-\Delta^2}, \quad n=0,\dots,N. \tag{33}$$

Hence, a dissipative system described by the following effective NHH (expressed via matrix form):

$$\boldsymbol{H}_\gamma^{\rm mb} = i\boldsymbol{M}_{N+1}^{\rm red}, \tag{34}$$

which is defined for $(N+1)$ modes $\vec{\gamma}'$, exhibits the EP of order $(N+1)$.

Therefore, by recursively repeating the same procedure, one can create new effective NHHs with higher-order EPs out of various evolution matrices, generated by the initial matrix $\boldsymbol{M}_A$ corresponding to a bimodal anti-$\mathcal{PT}$-symmetric cavity. Importantly, the same conclusions can be drawn for any two-mode linear $U(1)$-symmetric open systems exhibiting an EP, including those with coherent mode coupling.

We note, when constructing moments-based NHH, according to Eq. (34), one can always rescale the inner decoherence rate Γ to ensure that the achieved resolution of a given high-order EPs via coherence function and spectra is maximal.

VI. DISCUSSION AND CONCLUSIONS

In this article, we have proposed a method to properly define a new class of NHHs (so-called moments-based NHH) for quadratic Liouvillian systems, and which can exhibit higher-order quantum EPs. These moments-based NHHs are derived via a quantization of system operator moments obtained via a given quadratic Liouvillian. This approach correctly captures the dynamics of high-order system operators without semiclassical approximations, i.e., including the effect of quantum jumps [57, 59]. In other words, we have proposed a simple and effective method for engineering higher-order quantum EPs based on the moments-based NHH.

The dynamics of the moments-based NHHs can be associated with networks of dissipative coupled resonators. That is, by starting from a quadratic Liouvillian, describing, e.g., a two-mode system, the resulting evolution matrices, governing the higher-order field moments, can be cast to the new NHHs, which can, thus, describe networks of coupled cavities instead. To put it another way, one can assign to moments-based NHHs a clear physical meaning by mapping them onto the Liouvillian dynamics of the high-order moments of coupled resonators. This can be prove also useful to design lattice of $U(1)$ resonators with higher-order EPs, which could have applications for transport properties [92].

Compared to other existing methods for constructing NHHs with high-order EPs [53, 55, 56], the main advantage of our approach lies in its *simplicity and preservation of the commensurate character of mode coupling strengths*. Another outcome of the method developed here is to reveal the rich structure of quadratic Liouvillians for the dynamics of higher-order moments of the system operators, which can, thus, be probed by coherence and spectral functions of any order.

As an example, we have analyzed a $U(1)$-symmetric cavity with incoherent mode coupling, which can also possess the anti-$\mathcal{PT}$-symmetry. The system studied in Sec. V can serve as a promising platform for implementing structures with high-order EPs. Indeed, such systems with incoherent mode coupling can combine features related to both $\mathcal{PT}$ and anti-$\mathcal{PT}$-symmetries, as has been demonstrated in Ref. [63]. The construction of the physical moments-based NHH allows to obtain such EPs without building complex networks of coupled cavities, compared to the previous works [53–56]. Instead, one can just excite additional modes in a multimode cavity, ensuring that the system modes interact incoherently. Moreover, according to the previous section, the incoherent mode coupling strengths in such systems are commensurate and do not require fine tuning as, e.g., in Ref. [53], where coupling constants have incommensurate irrational prefactors, due to the expansion of the Hilbert space of an effective NHH. This conclusion is also valid when the intermode interaction is coherent.

The method proposed here allows to construct moments-based NHHs which can be both symmetric and asymmetric. Such asymmetrical mode coupling can be engineered via backscattering processes, as has been shown in Refs. [71, 72, 93]. Moreover, exploiting such asymmetric incoherent mode interactions, one can also implement a recently proposed scheme [56], which is based on the chiral nature of coherently coupled cavities, but rather using a much simpler *single multimode* cavity. Using our approach, one can also engineer block triangular moments-based NHHs highlighting the chirality in the mode coupling [56]. The latter can be obtained by combining Eq. (8) and Schur's triangularization theorem [94].

Apart from the interest in exploiting higher-order quantum Liouvillian EPs for constructing new moments-based NHHs, it also prompts the question of a further utility of such EPs, which can be revealed by coherence and spectral functions [63]. After all, it is exactly the Liouvillian eigenspace which correctly captures the dynamics of the fields, and therefore the moments-based NHH derived from it can be seem more physically justified.

The Liouvillian eigenspace genuinely incorporates the effect of quantum jumps, the *averaged* effect of which is captured in the dynamics of the field moments expressed via corresponding evolution matrices. As such, the mapping between the evolution matrices, governing the moments of the system operators, and moments-based NHHs ensures that the latter has a clear physical meaning also in the quantum limit. Note that the presence of quantum jumps can profoundly affect the system *dynamical* spectra, e.g., in some finite dimensional systems, the jumps can even lead to a shift of an EP in the parameter space [60].

The proposed method is universal in its nature. According to Eqs. (8) and (13), it can be applied to any physically realizable matrices $\boldsymbol{M}_\alpha$ and $\boldsymbol{M}_\beta$, each exhibiting an EP of some order, and which can be related to arbitrary quadratic Liouvillians. For instance, matrices $\boldsymbol{M}_\alpha$ and $\boldsymbol{M}_\beta$ can represent a Z_2-symmetric quadratic Liouvillian which describes a nonlinear parametric dissipative process.

We note that with the help of the described here technique, one can also analyze spectral degeneracies in systems with a weak nonlinearity, e.g., with Kerr-like interactions between modes. In that case, one can still invoke the Gaussian approximation for the fields fluctuations near the steady state, where the fields intensities are assumed to be fixed. As such, one can analytically derive and experimentally probe the system dynamical critical points by means of higher-order coherence and spectral functions of the fields fluctuations even in the presence of weak nonlinearities.

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