Flattening the Curve with Einstein's Quantum Elevator: Hermitization of Non-Hermitian Hamiltonians via the Vielbein Formalism

Chia-Yi Ju,[1] Adam Miranowicz,[2,3] Fabrizio Minganti,[3]
Chuan-Tsung Chan,[4,*] Guang-Yin Chen,[1,†] and Franco Nori[3,5,6]

[1] *Department of Physics, National Chung Hsing University, Taichung 402, Taiwan*
[2] *Institute of Spintronics and Quantum Information, Faculty of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland*
[3] *Theoretical Quantum Physics Laboratory, RIKEN Cluster for Pioneering Research, Wakoshi, Saitama 351-0198, Japan*
[4] *Department of Applied Physics, Tunghai University, Taichung 407, Taiwan*
[5] *RIKEN Center for Quantum Computing (RQC), Wakoshi, Saitama 351-0198, Japan*
[6] *Department of Physics, University of Michigan, Ann Arbor, Michigan 48109-1040, USA*

The non-triviality of Hilbert space geometries in non-Hermitian quantum systems sometimes blurs the underlying physics. We present a systematic study of the vielbein formalism which transforms the Hilbert spaces of non-Hermitian systems into the conventional ones, rendering the induced Hamiltonian to be Hermitian. In other words, any non-Hermitian Hamiltonian can be "transformed" into a Hermitian one without altering the physics. Thus, we show how to find a reference frame (corresponding to Einstein's quantum elevator) in which a non-Hermitian system, described by a non-trivial Hilbert space, reduces to a Hermitian system within the standard formalism of quantum mechanics for a Hilbert space.

Since the discovery of $\mathcal{PT}$-symmetric quantum mechanics [1–4], non-Hermitian quantum mechanics has become a very popular research field in quantum physics [5–8]. Even though the underlying mechanism of $\mathcal{PT}$-symmetric quantum mechanics was originally constructed from symmetry, many studies [9–13] have pointed out that the geometric information of the Hilbert space is hidden in that underlying mechanism. To be more precise, the dual space needs to be modified by a metric operator G, which often renders the Hilbert space geometry non-trivial or even dynamical.

This dynamics can be better understood with the help of Einstein's elevator gedanken experiment. This gedanken experiment of a free-falling elevator laid the theoretical foundations for general relativity by showing the equivalence between inertial reference frames in a uniform gravitational field (curved spacetime) and accelerating reference frames, in which physical phenomena can be described within the gravitation-free (locally flat spacetime) special-relativity [14].

We, therefore, ask an analogous question but concerning Hermitian and non-Hermitian formalisms of quantum mechanics: Does exist a reference frame (a quantum version of Einstein's elevator) in which a non-Hermitian system, described by a non-trivial Hilbert space, reduces to a Hermitian system, within the standard formalism of quantum mechanics for a "flat" Hilbert space? So the question is how to flatten the Hilbert space of a given non-Hermitian system. We constructively answer the question by applying the vielbein formalism [15]; namely, breaking the metric into two pieces and spreading those into the vector space and its dual space so that the system seems flat everywhere (see Fig. 1).

To be more precise, we can choose some non-coordinate bases [15], via the vielbein technique, to simplify the calculations and obtain new insights into the systems under study. Indeed, the vielbein formalism is useful in many fields of physics, including general relativity [16, 17], supergravity [18–20], superstring theories [21–24], etc. The main reason for applying the vielbein formalism is that it maps non-trivial space phenomena into a simpler space (and back).

Since the Hilbert spaces of non-Hermitian quantum systems are not trivial, it is useful to study the vielbeins in these systems. Although some rudimentary ideas regarding the vielbein formalism have been studied [25–28], we here provide a clearer geometric understanding of this formalism. With the vielbein formalism, the time evolution of the transformed states is always governed by an induced Hermitian Hamiltonian.

In addition to how the formalism works, we also study the relations between different choices of vielbeins. This leads to some gauge transformation [29–32] which does not affect the physics. With different choices of gauges, the states evolve with different induced Hermitian Hamiltonians. These induced Hamiltonians are, in some sense, an artifact that simplifies calculations but does not alter the final physical results.

A classical mechanics analogy of the gauge choice is a rotating or accelerating frame which causes a fictitious force. The induced Hamiltonian plays a similar role as those fictitious forces in the time-dependent frame (see Fig. 2).

In fact, the widely used Heisenberg and interaction pictures in Hermitian quantum mechanics are merely different choices of vielbeins. After the construction of the vielbein formalism in the Hilbert spaces of quantum states and its gauge symmetry, we show some examples of the vielbein formalism including how the Heisenberg and interaction pictures are related to the vielbeins.

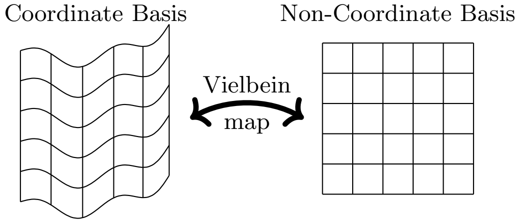

FIG. 1. An illustration of "flattening" the coordinates via the vielbein formalism. This procedure flattens the space (or curve in 1D). (Note that Regge calculus also flattens the curved manifold into flat space with deficit angles, that measure the local curvature [16].)

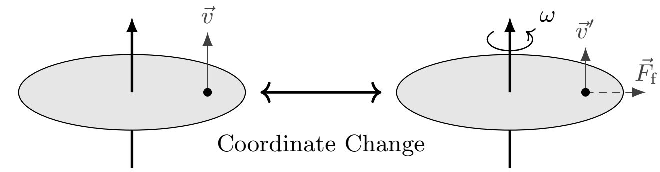

FIG. 2. Fictitious force induced from the coordinate change. This is a classical analog of different gauge choices inducing different Hamiltonians. As usual, there is no best gauge choice for all physical systems.

From a metric to a vielbein.—Unlike Hermitian quantum systems, where the inner product between two states in Hilbert space is the familiar $\langle\phi|\psi\rangle$, the Hilbert spaces of non-Hermitian quantum systems can have additional geometric structures so that the inner products in Hilbert space become $\langle\!\langle\phi|\psi\rangle\!\rangle = \langle\phi| G |\psi\rangle$, where $\langle\!\langle\phi| = \langle\phi| G$ is the corresponding dual state of $|\phi\rangle$ in the metricized Hilbert space with a metric G (see Table I for an explicit example). G has to be Hermitian and positive-definite for a proper Hilbert space.

In addition to the Hilbert space constraints mentioned above, this metric should also be constrained by the physics. The Hilbert space metric can be found by treating Schrödinger's equation as a parallel transport [13]. It has been shown that if the Schrödinger equation of a system is

$$\partial_t |\psi\rangle = -iH |\psi\rangle, \tag{1}$$

where H is its Hamiltonian, the compatibility of the metric G with the Schrödinger equation leads to

$$\partial_t G + iH^\dagger G - iGH = 0. \tag{2}$$

Although the solution for G is not unique, they all differ by a gauge transformation, $G \to G' = T^\dagger G T$, where T satisfies $\partial_t T + iHT - iTH = 0$.

Conventional	Metricized
$\langle\phi|\psi\rangle = \begin{pmatrix}\phi_1^* & \phi_2^*\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$	$\langle\!\langle\phi|\psi\rangle\!\rangle = \langle\phi| G |\psi\rangle$ $= \begin{pmatrix}\phi_1^* & \phi_2^*\end{pmatrix}\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$

TABLE I. Comparing the two inner products of two dimensional Hilbert spaces. In the conventional inner product, the dual state is just the Hermitian conjugate of the state; the dual state in the metricized space carries an additional metric operator G. Note that in the Hermitian case, the G can always be chosen to be the identity which reduces back to the conventional space [13].

Even though the metric carries the information of the Hilbert space geometry, like the cases mentioned previously, it is not always desirable to keep the metric explicitly. A systematic way of "removing" the metric is to adopt the vielbein formalism, so that vectors are "flat" everywhere.

Since G is Hermitian and positive-definite, it can be decomposed into

$$G = \mathcal{E}^\dagger \mathcal{E}, \tag{3}$$

where the operator $\mathcal{E}$ plays the role of a vielbein.

We can, therefore, use Eq. (3) to redefine the states in a locally Hermitian frame (in analogy to a locally flat frame). To be more specific, we define the state in the locally Hermitian frame to be

$$|\psi]] = \mathcal{E} |\psi\rangle. \tag{4}$$

It is obvious that the state evolution is no longer the same as Eq. (1). A simple calculation shows that the time evolution of the new state is

$$\partial_t |\psi]] = -iH_\flat |\psi]], \tag{5}$$

where

$$H_\flat = \mathcal{E} H \mathcal{E}^{-1} + i\left(\partial_t \mathcal{E}\right)\mathcal{E}^{-1} \tag{6}$$

is the induced Hamiltonian. A quick calculation shows that Eq. (6) guarantees that $H_\flat = H_\flat^\dagger$ (see the supplemental material [33] for a detailed derivation). This means that the induced Hamiltonian through the vielbein formalism, $H_\flat$, is always Hermitian.

The induced Hamiltonian being Hermitian implies that the dual state of $|\psi]]$ is the direct Hermitian conjugate of the state, i.e., $[[\psi| = (|\psi]])^\dagger = \langle\psi| \mathcal{E}^\dagger$. Hence, the inner product of the states in the locally Hermitian frame is reduced back to the usual inner product in the Hermitian system, while implicitly preserving the geometry of the Hilbert space, i.e.,

$$[[\psi_1|\psi_2]] = \langle\psi_1| G |\psi_2\rangle = \langle\!\langle\psi_1|\psi_2\rangle\!\rangle. \tag{7}$$

Therefore, the vielbein $\mathcal{E}$ transforms any non-Hermitian Hamiltonians to Hermitian ones. This is true even at exceptional points (EPs) [34–36].

	Original	Hermitianized
State and dual state	$\|\psi\rangle\rangle = \|\psi\rangle$ and $\langle\langle\psi\| = \langle\psi\| G$	$\|\psi]] = \mathcal{E}\|\psi\rangle$ and $[[\psi\| = \langle\psi\|\mathcal{E}^\dagger$
Inner product	$\langle\langle\phi\|\psi\rangle\rangle = \langle\phi\| G \|\psi\rangle$	$[[\phi\|\psi]] = \langle\phi\|\mathcal{E}^\dagger\mathcal{E}\|\psi\rangle$
Expectation values	$\langle\mathcal{O}\rangle = \langle\langle\psi\|\mathcal{O}\|\psi\rangle\rangle$	$\langle\mathcal{O}\rangle = [[\psi\|\mathcal{O}_\flat\|\psi]] = [[\psi\|\mathcal{E}\mathcal{O}\mathcal{E}^{-1}\|\psi]]$
Governing Equations	$\partial_t\|\psi\rangle\rangle = -iH\|\psi\rangle\rangle, \quad \partial_t\langle\langle\psi\| = i\langle\langle\psi\| H,$ $\partial_t G = i\left(GH - H^\dagger G\right)$	$\partial_t\|\psi]] = -iH_\flat\|\psi\rangle\rangle, \quad \partial_t [[\psi\| = i\langle\langle\psi\| H_\flat,$ $H_\flat = \mathcal{E}H\mathcal{E}^{-1} + i\left(\partial_t\mathcal{E}\right)\mathcal{E}^{-1}$

TABLE II. Relations between the original Hilbert space and the Hermitionized Hilbert space. Since the metric G is Hermitian and positive-definite, it can always be decomposed into $G = \mathcal{E}^\dagger\mathcal{E}$.

Observables.—The expectation value of an operator $\mathcal{O}$ is

$$\begin{aligned}\langle\mathcal{O}\rangle &= \langle\langle\psi|\mathcal{O}|\psi\rangle\rangle = \langle\psi| G\mathcal{O}|\psi\rangle \\ &= \langle\psi|\mathcal{E}^\dagger\mathcal{E}\mathcal{O}|\psi\rangle = [[\psi|\mathcal{E}\mathcal{O}\mathcal{E}^{-1}|\psi]]\,.\end{aligned} \tag{8}$$

This means that the operator $\mathcal{O}$ in the locally Hermitian frame becomes $\mathcal{O}_\flat = \mathcal{E}\mathcal{O}\mathcal{E}^{-1}$.

Using the fact that a self-adjoint operator $\mathcal{O}$ in the original space satisfies $\mathcal{O}^\dagger G = G\mathcal{O}$, together with Eq. (3) we find that

$$\mathcal{O}_\flat^\dagger = \left(\mathcal{E}^{-1}\right)^\dagger \mathcal{O}^\dagger\mathcal{E}^\dagger = \mathcal{E}\mathcal{O}\mathcal{E}^{-1} = \mathcal{O}_\flat, \tag{9}$$

i.e., the corresponding observable is also Hermitian in the locally Hermitian frame. Moreover, since $\mathcal{O}_\flat$ is merely a similarity transformation of $\mathcal{O}$, the eigenvalues of $\mathcal{O}_\flat$ are identical to the ones of $\mathcal{O}$.

Some comparisons between the original Hilbert space and the Hermitianized one are listed in Table II.

An additional gauge symmetry—The vielbein $\mathcal{E}$ is obtained from Eq. (3), hence, by construction there are some inherited gauge symmetries from the metric G. Nevertheless, in parallel to the case in differential geometry, the vielbein introduced here also has more gauge freedoms than the metric. The gauge transformation for the vielbein is a unitary transformation, i.e., $\mathcal{E} \to \mathcal{E}' = U\mathcal{E}$, where U is any unitary operator. This additional gauge choice comes from the invariance of G; to be more specific, $G' = \mathcal{E}'^\dagger\mathcal{E}' = \mathcal{E}^\dagger\mathcal{E} = G$.

Since U can be any unitary operator, it can be generated by $\partial_t U = -iH_\mathrm{L}U + iUH_\mathrm{R}$, where H_L and H_R are Hermitian operators with $U(t=0)$ being unitary. Then Eq. (6) for $\mathcal{E}'$ becomes

$$\begin{aligned}H'_\flat &= \mathcal{E}'H\mathcal{E}'^{-1} + i\left(\partial_t\mathcal{E}'\right)\mathcal{E}'^{-1} \\ &= H_\mathrm{L} + U\left(H_\flat - H_\mathrm{R}\right)U^{-1}.\end{aligned} \tag{10}$$

The detailed derivation can be found in the supplemental material [33].

This result not only shows that the induced Hamiltonian depends on the gauge choice, but it also shows that $H_\flat$ can be chosen freely. To be more specific, given an $\mathcal{E}_1$ which induces a Hamiltonian $H_{1\flat}$, we can make a gauge transformation, U_{21}, such that the states evolution is governed by $H_{2\flat}$ for U_{21} satisfying $\partial_t U_{21} = -iH_{2\flat}U_{21} + iU_{21}H_{1\flat}$.

This means that even though $H_\flat$ governs the dynamics of the corresponding $|\psi]]$, it is, in fact, telling us *how the gauge choice evolves with time without altering the physics.*

This gauge transformation might seem redundant at first; nevertheless, this has been often used in Hermitian quantum mechanics already. For example, the Heisenberg picture and interaction picture are special cases of the vielbein formalism with special choices of gauges ($H_\flat = 0$ and $H_\flat = H_\mathrm{I}(t)$ respectively). With this tool, we can "remove" the non-Hermiticity of the Hamiltonians.

Examples.—To show how the vielbein formalism works, some examples, both in Hermitian and non-Hermitian systems, are provided in the following.

Example 1: The Heisenberg picture and interaction pictures as gauge choices. The Heisenberg picture is historically the first picture of quantum mechanics extensively applied in many Hermitian studies. The main idea of the technique is to move all the time-dependence to the operator but leave the states time-independent. For the sake of clarity, we keep the time dependence of the states and operators explicit here.

To achieve this, one first finds a unitary operator $\mathcal{U}_\mathrm{H}(t)$, satisfying $\partial_t\mathcal{U}_\mathrm{H}(t) = -iH(t)\mathcal{U}_\mathrm{H}(t)$, and $\mathcal{U}_\mathrm{H}(0) = \mathbb{1}$. It is well-known that the states and the operators in the Heisenberg picture are defined to be $|\psi\rangle_\mathrm{H} = |\psi(0)\rangle$ and $\mathcal{O}_\mathrm{H}(t) = \mathcal{U}_\mathrm{H}^\dagger(t)\mathcal{O}(t)\mathcal{U}_\mathrm{H}(t)$, so that the operators carry all the time dependence while states have none, while leaving the physics unaltered, namely,

$$\langle\mathcal{O}\rangle(t) = \langle\psi(t)|\mathcal{O}(t)|\psi(t)\rangle = {}_\mathrm{H}\langle\psi|\mathcal{O}_\mathrm{H}(t)|\psi\rangle_\mathrm{H}\,. \tag{11}$$

To show that this is, in fact, a special case of the vielbein formalism, we let the vielbein be $\mathcal{E}(t) = \mathcal{U}_\mathrm{H}^{-1}(t) = \mathcal{U}_\mathrm{H}^\dagger(t)$. So that the states are

$$|\psi(t)]] = \mathcal{E}(t)|\psi(t)\rangle = \mathcal{U}_\mathrm{H}^{-1}(t)|\psi(t)\rangle = |\psi\rangle_\mathrm{H}\,, \tag{12}$$

and the observables are

$$\begin{aligned}\mathcal{O}_\flat(t) &= \mathcal{E}(t)\mathcal{O}(t)\mathcal{E}^{-1}(t) = \mathcal{U}_\mathrm{H}^{-1}(t)\mathcal{O}(t)\mathcal{U}_\mathrm{H}(t) \\ &= \mathcal{U}_\mathrm{H}^\dagger(t)\mathcal{O}(t)\mathcal{U}_\mathrm{H}(t) = \mathcal{O}_\mathrm{H}.\end{aligned} \tag{13}$$

The induced Hamiltonian is

$$H_\flat(t) = \mathcal{E}(t)H(t)\mathcal{E}^{-1}(t) + i\left(\partial_t\mathcal{E}(t)\right)\mathcal{E}^{-1}(t) = 0. \quad (14)$$

Hence, the Heisenberg picture is the same as choosing the vielbein satisfying $\mathcal{E}(0) = \mathbb{1}$, with the induced Hamiltonian $H_\flat = 0$.

The interaction picture, on the other hand, is a different gauge choice, where $H_\flat = H_{\mathrm{I}}(t)$. The detailed derivation can be found in the supplemental material [33].

Example 2: A non-Hermitian case.—Here we demonstrate how the vielbein formalism works using the following Hamiltonian [37]

$$H = \frac{\omega}{2}\sigma_x - i\frac{\gamma}{2}\sigma^+\sigma^-, \quad (15)$$

where $\omega \neq 0$ and $\widetilde{\gamma} = \gamma/\omega$. The Hilbert space in this example is finite dimensional, but it can still be used in infinite-dimensional Hilbert space cases as well (see the supplemental material [33] for an infinite dimensional example).

We split the discussion of this Hamiltonian into three cases [for $|\gamma| < 2|\omega|$, $|\gamma| > 2|\omega|$, and $\gamma = \pm 2\omega$], because the Hamiltonian is non-diagonalizable at $\gamma = \pm 2\omega$. In addition, we use three different starting points (the metric G, vielbein $\mathcal{E}$ for $H_\flat = 0$, and the vielbein $\mathcal{E}$ for $H_\flat \neq 0$) in these three cases to show that they are mathematically equivalent.

1. Case $|\gamma| < 2|\omega|$: We start with the metric method in this case. Solving Eq. (2) together with G being Hermitian and positive-definite, we find the metric

$$\begin{aligned} G = e^{\gamma t/2} &\begin{pmatrix} f^* & f \\ \left(-i\frac{\gamma}{2\omega} + \lambda_<\right) f^* & \left(-i\frac{\gamma}{2\omega} - \lambda_<\right) f \end{pmatrix} \\ &\times \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} f & \left(i\frac{\gamma}{2\omega} + \lambda_<\right) f \\ f^* & \left(i\frac{\gamma}{2\omega} - \lambda_<\right) f^* \end{pmatrix}, \end{aligned} \quad (16)$$

where $f = \exp\left(i\lambda_<\omega t/2\right)$, $\lambda_< = \left[1 - \gamma^2/(2\omega)^2\right]^{1/2}$, and the g_{ij}'s are constants, such that $g_{11} > 0$, $g_{22} > 0$, $g_{12}^* = g_{21}$, and $|g_{12}|^2 < g_{11}g_{22}$.

Using Eq. (3), we find the corresponding $\mathcal{E}$ being

$$\mathcal{E} = e^{\gamma t/4}\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} f & \left(i\frac{\gamma}{2\omega} + \lambda_<\right) f \\ f^* & \left(i\frac{\gamma}{2\omega} - \lambda_<\right) f^* \end{pmatrix}, \quad (17)$$

where $g_{ij} = \sum_k h_{ik}h_{kj}^*$. Note that the h_{ij}'s can be time-dependent functions despite the g_{ij}'s being constants.

Nevertheless, we first treat the h_{ij}'s as constants, using Eq. (6), and find the induced Hamiltonian $H_\flat = 0$, since $\partial_t\mathcal{E} = i\mathcal{E}H$.

We next make a gauge transformation to $\mathcal{E}' = U\mathcal{E}$, where

$$U = \exp\left(-i\frac{\omega t}{2}\sigma_x\right) = \begin{pmatrix} \cos\dfrac{\omega t}{2} & -i\sin\dfrac{\omega t}{2} \\ -i\sin\dfrac{\omega t}{2} & \cos\dfrac{\omega t}{2} \end{pmatrix}. \quad (18)$$

A direct calculation shows that the induced Hamiltonian of $\mathcal{E}'$ is then $H_\flat' = \omega\sigma_x/2$. So that if $\mathcal{E}'$ is chosen to be the vielbein, the state evolution is governed by $\partial_t\,|\psi]] = H_\flat'\,|\psi]]$.

2. Case $|\gamma| > 2|\omega|$: To find the corresponding metric G, we could calculate the metric using Eq. (2) like we did in the previous case. Nevertheless, this time we start with Eq. (6), while letting $H_\flat = 0$ so the general solution of the vielbein becomes

$$\mathcal{E} = e^{\gamma t/4}\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} f^+ & i\left(\frac{\gamma}{2\omega} - \lambda_>\right) f^+ \\ f^- & i\left(\frac{\gamma}{2\omega} + \lambda_>\right) f^- \end{pmatrix}, \quad (19)$$

where $f^\pm = \exp\left(\pm\lambda_>\omega t/2\right)$, $\lambda_> = \left[\gamma^2/\left(4\omega^2\right) - 1\right]^{1/2}$ and the matrix of h_{ij}'s is constant with a non-vanishing determinant.

We can then use Eq. (3) to find that

$$\begin{aligned} G = e^{\gamma t/2} &\begin{pmatrix} f^+ & f^- \\ -i\left(\frac{\gamma}{2\omega} - \lambda_>\right) f^+ & -i\left(\frac{\gamma}{2\omega} + \lambda_>\right) f^- \end{pmatrix} \\ &\times \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} f^+ & i\left(\frac{\gamma}{2\omega} - \lambda_>\right) f^+ \\ f^- & i\left(\frac{\gamma}{2\omega} + \lambda_>\right) f^- \end{pmatrix}, \end{aligned} \quad (20)$$

where $g_{ij} = \sum_k h_{ik}h_{kj}^*$. Now the g_{ij}'s are constants, such that $g_{11} > 0$, $g_{22} > 0$, $g_{12}^* = g_{21}$, and $|g_{12}|^2 < g_{11}g_{22}$, which indeed make the metric Hermitian and positive-definite.

We can, again, apply a gauge transformation from $\mathcal{E}$ to $\mathcal{E}' = U\mathcal{E}$, with $U = \exp\left[(-i/2)\omega t\sigma_x\right]$. Direct calculation shows that the induced Hamiltonian also becomes $H_\flat' = \omega\sigma_x/2$.

3. Exceptional point at $\gamma = \pm 2\omega$: Now

$$H_{\mathrm{EP}} = \frac{\omega}{2}\begin{pmatrix} \mp 2i & 1 \\ 1 & 0 \end{pmatrix}, \quad (21)$$

which is non-diagonalizable and corresponds to an EP [36]. Although we can still find its corresponding metric by solving Eq. (2) directly, we start with Eq. (6) while setting $H_\flat = \omega\sigma_x/2$. Thus, using Eq. (6), we find

$$\mathcal{E} = e^{\frac{(\pm 1 - i\sigma_x)\omega t}{2}}\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} 1 & \pm i \\ 2\omega t & i(\pm 2\omega t - 4) \end{pmatrix}, \quad (22)$$

where the constant matrix made of h_{ij}'s has a non-vanishing determinant.

Using Eq. (3), we, again, find the corresponding metric

$$\begin{aligned} G = e^{\pm\omega t} &\begin{pmatrix} 1 & 2\omega t \\ \mp i & -i(\pm 2\omega t - 4) \end{pmatrix}\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \\ &\times \begin{pmatrix} 1 & \pm i \\ 2\omega t & i(\pm 2\omega t - 4) \end{pmatrix}, \end{aligned} \quad (23)$$

where $g_{ij} = \sum_k h_{ik}h_{kj}^*$. Clearly we can always use $U = \exp\left(i\omega t\sigma_x/2\right)$ to transform the induced Hamiltonian into $H_\flat' = 0$.

Conclusion.—The non-trivial Hilbert space geometric structures in non-Hermitian quantum systems sometimes complicate the physical description of these systems. Following the geometrical meaning of Schrödinger's equation, we find that non-Hermitian Hamiltonians can be transformed into Hermitian ones via the vielbein formalism shedding a new light on the physics of non-Hermitian systems. We also present a systematic study on an additional gauge symmetry which originates from the freedom of choosing vielbein frames, where the quantum states evolution is described by different Hamiltonians. Furthermore, the vielbein formalism is *not* restricted to non-Hermitian quantum systems. The gauge freedoms in the vielbein formalism in Hermitian systems also grants us the freedom to choose frames, the Heisenberg and interaction pictures for example, that are easier to work with.

Acknowledgment.—C.Y.J. is supported by the Ministry of Science and Technology (MOST), Taiwan, through Grant No. MOST 110-2811-M-005-518. G.Y.C. is partially supported by the National Center for Theoretical Sciences, and the MOST through Grant No. MOST 110-2123-M-006-001. Both C.Y.J. and G.Y.C. are supported partially by the MOST through Grant No. MOST 110-2112-M-005-002. C.T.C. is partially supported by the MOST through Grant No. MOST 109-2112-M-029-006- and 110-2112-M-029-006-. A.M. is supported by the Polish National Science Centre (NCN) under the Maestro Grant No. DEC-2019/34/A/ST2/00081. F.N. is supported in part by: Nippon Telegraph and Telephone Corporation (NTT) Research, the Japan Science and Technology Agency (JST) [via the Quantum Leap Flagship Program (Q-LEAP), the Moonshot R&D Grant Number JPMJMS2061, and the Centers of Research Excellence in Science and Technology (CREST) Grant No. JPMJCR1676], the Japan Society for the Promotion of Science (JSPS) [via the Grants-in-Aid for Scientific Research (KAKENHI) Grant No. JP20H00134 and the JSPS–RFBR Grant No. JPJSBP120194828], the Army Research Office (ARO) (Grant No. W911NF-18-1-0358), the Asian Office of Aerospace Research and Development (AOARD) (via Grant No. FA2386-20-1-4069), and the Foundational Questions Institute Fund (FQXi) via Grant No. FQXi-IAF19-06.

* ctchan@go.thu.edu.tw
† gychen@phys.nchu.edu.tw

[1] C. M. Bender and S. Boettcher, *Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry*, Phys. Rev. Lett. **80**, 5243 (1998).
[2] C. M. Bender, D. C. Brody, and H. F. Jones, *Complex extension of quantum mechanics*, Phys. Rev. Lett. **89**, 270401 (2002).
[3] C. M. Bender, J. Brod, A. Refig, and M. E. Reuter, *The $\mathcal{C}$ operator in $\mathcal{PT}$-symmetric quantum theories*, J. Phys A: Math. Gen. **37**, 10139 (2004).
[4] C. M. Bender, *Making sense of non-Hermitian Hamiltonians*, Rep. Prog. Phys. **70**, 947 (2007).
[5] D. C. Brody, *Consistency of PT-symmetric quantum mechanics*, J. Phys. A: Math. Theor. **49**, 10LT03 (2016).
[6] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, and D. N. Christodoulides, *Non-Hermitian physics and PT symmetry*, Nat. Phys. **14**, 11 (2018).
[7] F. Bagarello, R. Passante, and C. Trapani, eds., *Non-Hermitian Hamiltonians in Quantum Physics* (Springer International Publishing, 2016).
[8] Y. Ashida, Z. Gong, and M. Ueda, *Non-Hermitian physics*, Adv. Phys. **69**, 249 (2020).
[9] A. Mostafazadeh, *Pseudo-Hermiticity and generalized $\mathcal{PT}$- and $\mathcal{CPT}$-symmetries*, J. Math. Phys. **44**, 974 (2003).
[10] A. Mostafazadeh, *Time dependent Hilbert spaces, geometric phases, and general covariance in quantum mechanics*, Phys. Lett. A **320**, 375 (2004).
[11] D. C. Brody, *Biorthogonal quantum mechanics*, J. Phys. A: Math. Theor. **47**, 035305 (2013).
[12] M. Znojil, *Is $\mathcal{PT}$-symmetric quantum theory false as a fundamental theory?*, Acta Polytechnica (Prague) **56**, 254 (2016).
[13] C.-Y. Ju, A. Miranowicz, G.-Y. Chen, and F. Nori, *Non-Hermitian Hamiltonians and no-go theorems in quantum information*, Phys. Rev. A **100**, 062118 (2019).
[14] A. Einstein, *The Meaning of Relativity: Including the Relativistic Theory of the Non-Symmetric Field*, 5th ed. (Princeton Univers. Press, 2014).
[15] M. Nakahara, *Geometry, Topology and Physics*, 2nd ed. (IOP Publishing, Bristol, 2003).
[16] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Princeton Univers. Press, 2017).
[17] R. M. Wald, *General Relativity* (University of Chicago Press, 1984).
[18] P. van Nieuwenhuizen, *Supergravity*, Phys. Rep. **68**, 189 (1981).
[19] P. West, *Introduction to Supersymmetry and Supergravity* (World Scientific, 1990).
[20] W. Siegel, *Two-vierbein formalism for string-inspired axionic gravity*, Phys. Rev. D **47**, 5453 (1993).
[21] W. Siegel, *Superspace duality in low-energy superstrings*, Phys. Rev. D **48**, 2826 (1993).
[22] M. Poláček and W. Siegel, *T-duality off shell in 3D type II superspace*, J. High Energy Phys. **2014**, 107 (2014).
[23] C.-Y. Ju and W. Siegel, *Gauging unbroken symmetries in F-theory*, Phys. Rev. D **94**, 106004 (2016).
[24] W. D. Linch and W. Siegel, *F-theory from fundamental five-branes*, J. High Energy Phys. **2021**, 47 (2021).
[25] M. Znojil, *Time-dependent version of crypto-Hermitian quantum theory*, Phys. Rev. D **78**, 085003 (2008).
[26] A. Mostahazadeh, *Pseudo-Hermitian Reresentation of Quantum Mechanics*, Int. J. Geom. Methods Mod. Phys. **07**, 1191 (2010).
[27] R. Hamazaki, K. Kawabata, N. Kura, and M. Ueda, *Universality classes of non-Hermitian random matrices*, Phys. Rev. Res. **2**, 023286 (2020).
[28] T. Ohlsson and S. Zhou, *Density-matrix formalism for $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians with the Lindblad equation*, Phys. Rev. A **103**, 022218 (2021).
[29] O. D. Stefano, A. Settineri, V. Macrì, L. Garziano, R. Stassi, S. Savasta, and F. Nori, *Resolution of gauge ambiguities in ultrastrong-coupling cavity quantum elec-*

trodynamics, Nat. Phys. **15**, 803 (2019).

[30] L. Garziano, A. Settineri, O. D. Stefano, S. Savasta, and F. Nori, *Gauge invariance of the Dicke and Hopfield models*, Phys. Rev. A **102**, 023718 (2020).

[31] A. Settineri, O. D. Stefano, D. Zueco, S. Hughes, S. Savasta, and F. Nori, *Gauge freedom, quantum measurements, and time-dependent interactions in cavity QED*, Phys. Rev. Res. **3**, 023079 (2021).

[32] S. Savasta, O. D. Stefano, A. Settineri, D. Zueco, S. Hughes, and F. Nori, *Gauge principle and gauge invariance in two-level systems*, Phys. Rev. A **103**, 053703 (2021).

[33] See Supplemental Material below for the Hermiticity of the induced Hamiltonians, gauge choices of the induced Hamiltonians, the interaction picture as a gauge choice, and an infinite dimensional example.

[34] T. Kato, *Perturbation theory for linear operators*, 2nd ed., Grundlehren der mathematischen Wissenschaften: a series of comprehensive studies in mathematics (Springer, Berlin, 1976).

[35] W. D. Heiss, *Exceptional points of non-Hermitian operators*, J. Phys A: Math. Gen. **37**, 2455 (2004).

[36] Ş. K. Özdemir, S. Rotter, F. Nori, and L. Yang, *Parity–time symmetry and exceptional points in photonics*, Nat. Mater. **18**, 783 (2019).

[37] F. Minganti, A. Miranowicz, R. W. Chhajlany, and F. Nori, *Quantum exceptional points of non-Hermitian Hamiltonians and Liouvillians: The effects of quantum jumps*, Phys. Rev. A **100**, 062131 (2019).

Supplemental Material for "Flattening the Curve with Einstein's Quantum Elevator: Hermitization of Non-Hermitian Hamiltonians via the Vielbein Formalism"

We first briefly summarize the idea flow of the vielbein formalism and some equations that will be used in this supplemental material, so the reader does not have to switch back and forth between the main article and the supplemental material. Nevertheless, the reader should consult the main article for motivation and reasoning.

The flow line of the vielbein formalism can be summarized as follows:

1. The inner product between states is
$$\langle\langle \phi|\psi\rangle\rangle = \langle\phi|\, G\, |\psi\rangle\,, \tag{S1}$$
where G is the metric operator that is positive-definite and Hermitian.

2. The metric operator G can be decomposed into
$$G = \mathcal{E}^{\dagger}\mathcal{E}, \tag{S2}$$
where E is the vielbein.

3. Define the states in the "locally Hermitian frame" as
$$|\psi]] = \mathcal{E}\,|\psi\rangle\,, \tag{S3}$$
where the time evolution of the states become
$$\partial_t\,|\psi]] = -iH_\flat\,|\psi]]\,. \tag{S4}$$

4. The induced Hamiltonian $H_\flat$ originates from
$$H_\flat = \mathcal{E}H\mathcal{E}^{-1} + i\left(\partial_t\mathcal{E}\right)\mathcal{E}^{-1}. \tag{S5}$$

This supplemental material includes: 1) the detailed derivation of $H_\flat = H_\flat^\dagger$; 2) showing that $H_\flat$ can be chosen freely and is related to the gauge choice; 3) the interaction picture as a gauge choice; 4) the vielbein formalism on an infinite dimensional Hilbert space.

I. THE HERMITICITY OF THE INDUCED HAMILTONIAN

The goal of this section is to show that the induced Hamiltonian in the vielbein formalism is always Hermitian.

To show this, we use the fact that the time evolution equation of the metric G is
$$\partial_t G = i\left(GH - H^\dagger G\right). \tag{S6}$$

a direct calculation shows that
$$\begin{aligned}
H_\flat^\dagger &= \left(\mathcal{E}^\dagger\right)^{-1} H^\dagger\mathcal{E}^\dagger - i\left(\mathcal{E}^\dagger\right)^{-1}\left(\partial_t\mathcal{E}^\dagger\right)\\
&= \left(\mathcal{E}^\dagger\right)^{-1} H^\dagger\mathcal{E}^\dagger\left(\mathcal{E}\mathcal{E}^{-1}\right) - i\left(\mathcal{E}^\dagger\right)^{-1}\left(\partial_t\mathcal{E}^\dagger\right)\left(\mathcal{E}\mathcal{E}^{-1}\right)\\
&= \left(\mathcal{E}^\dagger\right)^{-1} H^\dagger G\mathcal{E}^{-1} - i\left(\mathcal{E}^\dagger\right)^{-1}\left(\partial_t\mathcal{E}^\dagger\mathcal{E}\right)\mathcal{E}^{-1} + i\left(\mathcal{E}^\dagger\right)^{-1}\mathcal{E}^\dagger\left(\partial_t\mathcal{E}\right)\mathcal{E}^{-1}\\
&= \left(\mathcal{E}^\dagger\right)^{-1} H^\dagger G\mathcal{E}^{-1} - i\left(\mathcal{E}^\dagger\right)^{-1}\left(\partial_t G\right)\mathcal{E}^{-1} + i\left(\partial_t\mathcal{E}\right)\mathcal{E}^{-1}\\
&= \left(\mathcal{E}^\dagger\right)^{-1} GH\mathcal{E}^{-1} + i\left(\partial_t\mathcal{E}\right)\mathcal{E}^{-1}\\
&= \mathcal{E}H\mathcal{E}^{-1} + i\left(\partial_t\mathcal{E}\right)\mathcal{E}^{-1} = H_\flat,
\end{aligned} \tag{S7}$$

where Eq. (S2) and Eq. (S6) are applied in the derivation.

Note that the gauge choice of vielbein $\mathcal{E}$ has not been specified. That is to say, any vielbein gauge choice renders $H_\flat = H_\flat^\dagger$. Hence the induced Hamiltonian via the vielbein formalism is always Hermitian.

II. THE GAUGE TRANSFORMATION ON $H_\flat$

This section focuses on the detailed proof of Eq. (10) in the main text and its implication.

It is know that the decomposition of Eq. (S2) is far from unique. If $\mathcal{E}$ satisfies Eq. (S2), we can always find another vielbein $\mathcal{E}' = U\mathcal{E}$ that also satisfies $G = \mathcal{E}'^\dagger \mathcal{E}'$. By construction, we find

$$\mathcal{E}^\dagger \mathcal{E} = G = \mathcal{E}'^\dagger E' = \mathcal{E} U^\dagger U E \tag{S8}$$

$$\Rightarrow U^\dagger U = \mathbb{1}. \tag{S9}$$

This shows that U can be any unitary operator. Therefore, the time derivative of U can always be written as

$$\partial_t U = -iH_\mathrm{L} U + iUH_\mathrm{R}, \tag{S10}$$

where H_L and H_R are Hermitian operators with $U(t=0)$ being unitary. Using Eq. (S5) for $\mathcal{E}'$, we find

$$\begin{aligned} H'_\flat &= E' H \mathcal{E}'^{-1} + i\left(\partial_t E'\right)\mathcal{E}'^{-1} \\ &= U\mathcal{E}H\mathcal{E}^{-1}U^{-1} + i\left(\partial_t U\mathcal{E}\right)\mathcal{E}^{-1}U^{-1} \\ &= U\mathcal{E}H\mathcal{E}^{-1}U^{-1} + H_\mathrm{L} U\mathcal{E}\mathcal{E}^{-1}U^{-1} \\ &\quad - UH_\mathrm{R}\mathcal{E}\mathcal{E}^{-1}U^{-1} + iU\left(\partial_t \mathcal{E}\right)\mathcal{E}^{-1}U^{-1} \\ &= H_\mathrm{L} + U\left(H_\flat - H_\mathrm{R}\right)U^{-1}, \end{aligned} \tag{S11}$$

which proves Eq. (10) in the main text.

This result shows that when we have a vielbein, say $\mathcal{E}_1$, that induces the Hamiltonian $H_{1\flat}$, we can apply a gauge transform on $\mathcal{E}_1$ to $\mathcal{E}_2 = U_{21}\mathcal{E}_1$, such that the new induced Hamiltonian $H_{2\flat}$ can be any given Hermitian operator. To achieve this, U_{21} needs to be unitary at some time t and its time derivative must satisfy

$$\partial_t U_{21} = -iH_{2\flat} U_{21} + iU_{21} H_{1\flat}. \tag{S12}$$

Hence, we can always choose a frame that is convenient to work with.

III. THE INTERACTION PICTURE AS A GAUGE CHOICE

Besides the Heisenberg picture shown in the main text, another standard picture in Hermitian quantum mechanics is the interaction picture, which is particularly useful in perturbation methods. We keep the time-dependence explicit in this section to avoid possible confusions.

The main idea of the interaction picture is to split the Hamiltonian into a "system" part and an "interaction" part, namely,

$$H(t) = H_\mathrm{s}(t) + H_\mathrm{int}(t), \tag{S13}$$

where $H_\mathrm{s}(t)$ and $H_\mathrm{int}(t)$ are Hermitian.

One then finds an $\mathcal{U}_\mathrm{I}(t)$ such that

$$\partial_t \mathcal{U}_\mathrm{I}(t) = -iH_\mathrm{s}(t)\mathcal{U}_\mathrm{I}(t) \tag{S14}$$

with $\mathcal{U}_\mathrm{I}(0) = \mathbb{1}$. It is obvious that the $\mathcal{U}_\mathrm{I}(t)$ satisfying these two conditions is a unitary operator. The states and the operators in the interaction picture are defined to be

$$\left|\psi(t)\right\rangle_\mathrm{I} = \mathcal{U}_\mathrm{I}^{-1}(t)\left|\psi(t)\right\rangle, \tag{S15}$$

$$\mathcal{O}_\mathrm{I}(t) = \mathcal{U}_\mathrm{I}^\dagger(t)\mathcal{O}(t)\mathcal{U}_\mathrm{I}(t). \tag{S16}$$

It can be shown that the time evolution of the states in the interaction picture is

$$i\partial_t \left|\psi(t)\right\rangle_\mathrm{I} = H_\mathrm{I}(t)\left|\psi(t)\right\rangle_\mathrm{I}, \tag{S17}$$

where

$$H_\mathrm{I}(t) = \mathcal{U}_\mathrm{I}^{-1}(t) H_\mathrm{int}(t)\mathcal{U}_\mathrm{I}(t). \tag{S18}$$

Back to the vielbein formalism, we can choose the vielbein to be

$$\mathcal{E}(t) = \mathcal{U}_{\mathrm{I}}^{-1}(t) = \mathcal{U}_{\mathrm{I}}^{\dagger}(t), \tag{S19}$$

so that the states and the operators become

$$|\psi(t)]] = \mathcal{E}(t)\,|\psi(t)\rangle = \mathcal{U}_{\mathrm{I}}^{-1}(t)\,|\psi(t)\rangle = |\psi\rangle_{\mathrm{I}}\,, \tag{S20}$$

$$\begin{aligned} \mathcal{O}_\flat(t) &= \mathcal{E}(t)\mathcal{O}(t)\mathcal{E}^{-1}(t) = \mathcal{U}_{\mathrm{I}}^{-1}(t)\mathcal{O}(t)\mathcal{U}_{\mathrm{I}}(t) \\ &= \mathcal{U}_{\mathrm{I}}^{\dagger}(t)\mathcal{O}(t)\mathcal{U}_{\mathrm{I}}(t) = \mathcal{O}_{\mathrm{I}}. \end{aligned} \tag{S21}$$

The induced Hamiltonian in this case is

$$\begin{aligned} H_\flat(t) &= \mathcal{E}(t)H(t)\mathcal{E}^{-1}(t) + i\left(\partial_t\mathcal{E}(t)\right)\mathcal{E}^{-1}(t) \\ &= \mathcal{U}_{\mathrm{I}}^{-1}(t)H(t)\mathcal{U}_{\mathrm{I}}(t) - i\mathcal{U}_{\mathrm{I}}^{-1}(t)\partial_t\mathcal{U}_{\mathrm{I}}(t) \\ &= \mathcal{U}_{\mathrm{I}}^{-1}(t)H(t)\mathcal{U}_{\mathrm{I}}(t) - \mathcal{U}_{\mathrm{I}}^{-1}(t)H_{\mathrm{s}}(t)\mathcal{U}_{\mathrm{I}}(t) \\ &= \mathcal{U}_{\mathrm{I}}^{-1}(t)H_{\mathrm{int}}(t)\mathcal{U}_{\mathrm{I}}(t) \\ &= H_{\mathrm{I}}(t). \end{aligned} \tag{S22}$$

Therefore, the interaction picture in Hermitian quantum mechanics is also a special choice of the vielbein.

IV. AN INFINITE DIMENSION NON-HERMITIAN SYSTEM EXAMPLE

In this subsection, we demonstrate that this vielbein formalism also works for a infinite dimensional Hilbert space with the Hamiltonian

$$\begin{aligned} H =& -i\frac{\gamma_a}{2}a^\dagger a - i\frac{\gamma_b}{2}b^\dagger b + g\left(a^\dagger b + b^\dagger a\right) \\ =& \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \begin{pmatrix} -i\dfrac{\gamma_a}{2} & g \\ g & -i\dfrac{\gamma_a}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \end{aligned} \tag{S23}$$

where a and b ($a^\dagger$ and $b^\dagger$) are the bosonic annihilation (creation) operators. For later convenience, we define the vacuum state $|0\rangle$ such that

$$a\,|0\rangle = |0\rangle \cdot 0 \;= 0, \tag{S24}$$

$$b\,|0\rangle = |0\rangle \cdot 0 \;= 0. \tag{S25}$$

A. Case Without Exceptional Point

When $|\gamma_a - \gamma_b| \neq 4|g|$, the Hamiltonian can be rewritten as

$$\begin{aligned} H &= h_+ c_+^{\mathrm{c}} c_+^{\mathrm{a}} + h_- c_-^{\mathrm{c}} c_-^{\mathrm{a}} \\ &= \begin{pmatrix} c_+^{\mathrm{c}} & c_-^{\mathrm{c}} \end{pmatrix} \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix} \begin{pmatrix} c_+^{\mathrm{a}} \\ c_-^{\mathrm{a}} \end{pmatrix}, \end{aligned} \tag{S26}$$

where

$$\begin{aligned} h_\pm =& -i\frac{\gamma_a + \gamma_b}{4} \pm \frac{\zeta}{4}, \\ c_\pm^{\mathrm{c}} =& \left[a^\dagger \mp \frac{\zeta \mp i\left(\gamma_a - \gamma_b\right)}{4g} b^\dagger\right], \\ c_\pm^{\mathrm{a}} =& \frac{1}{\zeta}\left[\frac{\zeta \pm i\left(\gamma_a - \gamma_b\right)}{2} a \mp 2gb\right], \\ \zeta^2 =& 16g^2 - \left(\gamma_a - \gamma_b\right)^2. \end{aligned} \tag{S27}$$

The commutation relations between H, c^{c}, and c^{a} are

$$\begin{aligned}\left[c_{\pm}^{\mathrm{a}}, c_{\pm}^{\mathrm{c}}\right] &= 1,\\ \left[c_{\pm}^{\mathrm{a}}, c_{\mp}^{\mathrm{c}}\right] &= 0,\\ \left[H, c_{\pm}^{\mathrm{c}}\right] &= h_{\pm} c_{\pm}^{\mathrm{c}},\\ \left[H, c_{\pm}^{\mathrm{a}}\right] &= -h_{\pm} c_{\pm}^{\mathrm{a}}.\end{aligned} \tag{S28}$$

Then by solving Eq. (S6), together with $G = G^{\dagger}$ and positive-definiteness, we find

$$G = \sum_{\substack{n_+=0\\ n_-=0}}^{\infty} \frac{g_{n_+n_-}}{(n_+!)^2 (n_-!)^2} \exp\left[-2t\left(n_+ \operatorname{Im} h_+ + n_- \operatorname{Im} h_-\right)\right] \left(c_-^{\mathrm{a}\dagger}\right)^{n_-} \left(c_+^{\mathrm{a}\dagger}\right)^{n_+} |0\rangle\langle 0| \left(c_+^{\mathrm{a}}\right)^{n_+} \left(c_-^{\mathrm{a}}\right)^{n_-}, \tag{S29}$$

where $g_{ij} > 0$ are constants.

To find the corresponding vielbeins, we only need to find one $\mathcal{E}$ that satisfies Eq. (S2); then, the general solution can be found by a simple gauge transformation. A vielbein that satisfies Eq. (S2) is

$$\mathcal{E} = \sum_{\substack{n_+=0\\ n_-=0}}^{\infty} \frac{h_{n_+n_-}}{(n_+!)^{3/2} (n_-!)^{3/2}} \exp\left[it\left(n_+ h_+ + n_- h_-\right)\right] \left(a^{\dagger}\right)^{n_-} \left(b^{\dagger}\right)^{n_+} |0\rangle\langle 0| \left(c_+^{\mathrm{a}}\right)^{n_+} \left(c_-^{\mathrm{a}}\right)^{n_-}, \tag{S30}$$

where $h_{n_+n_-}$'s are non-zero constants, and that $g_{n_+n_-} = \left|h_{n_+n_-}\right|^2 > 0$. A direct calculation shows that the induced Hamiltonian is $H_\flat = 0$.

We can make a gauge transformation on $\mathcal{E}$ to $\mathcal{E}'$ by

$$\mathcal{E}' = U\mathcal{E}, \tag{S31}$$

where

$$U = \sum_{\substack{m=0\\ n=0}}^{\infty} \frac{\exp\left[itg(m-n)\right]}{\sqrt{2}^{m+n} (m!)(n!)} \times \left(a^{\dagger} + b^{\dagger}\right)^{m} \left(a^{\dagger} - b^{\dagger}\right)^{n} |0\rangle\langle 0| a^m b^n, \tag{S32}$$

Then the induced Hermitian Hamiltonian becomes

$$H_\flat' = g(a^{\dagger} b + a b^{\dagger}), \tag{S33}$$

if the vielbein is chosen to be $\mathcal{E}'$ in Eq. (S31), namely,

$$\mathcal{E}' = \sum_{\substack{n_+=0\\ n_-=0}}^{\infty} \frac{h_{n_+n_-} \exp\left\{it\left[n_+\left(h_+ + g\right) + n_-\left(h_- - g\right)\right]\right\}}{\sqrt{2}^{n_+ + n_-} (n_+!)^{3/2} (n_-!)^{3/2}} \left(a^{\dagger} + b^{\dagger}\right)^{n_+} \left(a^{\dagger} - b^{\dagger}\right)^{n_-} |0\rangle\langle 0| \left(c_+^{\mathrm{a}}\right)^{n_+} \left(c_-^{\mathrm{a}}\right)^{n_-}. \tag{S34}$$

B. Exceptional point (EP) at $|\gamma_a - \gamma_b| = 4|g|$

When $\gamma_a - \gamma_b = 4\chi g$, where $\chi = \pm 1$, the Hamiltonian is at an EP and the Hamiltonian becomes

$$H = g\left[-i\left(\chi + \delta\right) a^{\dagger} a + i\left(\chi - \delta\right) b^{\dagger} b + a^{\dagger} b + b^{\dagger} a\right] \tag{S35}$$

$$= g\begin{pmatrix} a^{\dagger} & b^{\dagger}\end{pmatrix} \begin{pmatrix} -i\left(\chi + \delta\right) & 1 \\ 1 & i\left(\chi - \delta\right)\end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{S36}$$

where $\delta = \Delta\gamma/g$, $\gamma_a = 2\chi g + 2\Delta\gamma$ and $\gamma_b = -2\chi g + 2\Delta\gamma$. With a simple recombination of the operators, the Hamiltonian becomes

$$H = g\begin{pmatrix} d_+^{\mathrm{c}} & d_-^{\mathrm{c}}\end{pmatrix} \begin{pmatrix} i\Delta\gamma & 2i \\ 0 & i\Delta\gamma\end{pmatrix} \begin{pmatrix} d_+^{\mathrm{a}} \\ d_-^{\mathrm{a}} \end{pmatrix} \tag{S37}$$

$$= -i\Delta\gamma\left(d_+^{\mathrm{c}} d_+^{\mathrm{a}} + d_-^{\mathrm{c}} d_-^{\mathrm{a}}\right) + 2igd_+^{\mathrm{c}} d_-^{\mathrm{a}}, \tag{S38}$$

where

$$d_{\pm}^{\mathrm{c}} = \frac{1}{\sqrt{2}}\left(a^{\dagger} \mp i\chi b^{\dagger}\right), \quad d_{\pm}^{\mathrm{a}} = \frac{1}{\sqrt{2}}\left(a \pm i\chi b\right). \tag{S39}$$

The commutation relations in this case become

$$\begin{aligned} \left[d_{\pm}^{\mathrm{a}}, d_{\pm}^{\mathrm{c}}\right] &= 1, \\ \left[d_{\pm}^{\mathrm{a}}, d_{\mp}^{\mathrm{c}}\right] &= 0, \\ \left[H, d_{+}^{\mathrm{c}}\right] &= -i\Delta\gamma d_{+}^{\mathrm{c}}, \\ \left[H, d_{-}^{\mathrm{c}}\right] &= -i\Delta\gamma d_{-}^{\mathrm{c}} + 2igd_{+}^{\mathrm{c}}, \\ \left[H, d_{+}^{\mathrm{a}}\right] &= i\Delta\gamma d_{+}^{\mathrm{a}} - 2igd_{-}^{\mathrm{a}}, \\ \left[H, d_{-}^{\mathrm{a}}\right] &= i\Delta\gamma d_{-}^{\mathrm{a}}. \end{aligned} \tag{S40}$$

We can, again, use Eq. (S5) with $H_{\flat} = 0$ to find the corresponding vielbein,

$$\mathcal{E} = \sum_{\substack{n_+=0 \\ n_-=0}}^{\infty} \frac{h_{n_+n_-}}{(n_+!)^{3/2}(n_-!)^{3/2}} \exp\left[\Delta\gamma t\left(n_+ + n_-\right)\right]\left(a^{\dagger}\right)^{n_-}\left(b^{\dagger}\right)^{n_+}|0\rangle\langle 0|\left(d_{+}^{\mathrm{a}} - 2gtd_{-}^{\mathrm{a}}\right)^{n_+}\left(d_{-}^{\mathrm{a}}\right)^{n_-}, \tag{S41}$$

where the $h_{n_+n_-}$'s are nonzero constants. Therefore, the metric becomes

$$G = \sum_{\substack{n_+=0 \\ n_-=0}}^{\infty} \frac{g_{n_+n_-}}{(n_+!)^{2}(n_-!)^{2}} \exp\left[2\Delta\gamma t\left(n_+ + n_-\right)\right]\left(d_{-}^{\mathrm{a}\dagger}\right)^{n_-}\left(d_{+}^{\mathrm{a}\dagger} - 2gtd_{-}^{\mathrm{a}\dagger}\right)^{n_+}|0\rangle\langle 0|\left(d_{+}^{\mathrm{a}} - 2gtd_{-}^{\mathrm{a}}\right)^{n_+}\left(d_{-}^{\mathrm{a}}\right)^{n_-}, \tag{S42}$$

where $g_{n_+n_-} = \left|h_{n_+n_-}\right|^2$. Once again, we can apply a transformation $\mathcal{E} \to \mathcal{E}' = U\mathcal{E}$, where

$$U = \sum_{\substack{m=0 \\ n=0}}^{\infty} \frac{\exp\left[itg(m-n)\right]}{\sqrt{2}^{m+n}(m!)(n!)}\left(a^{\dagger} + b^{\dagger}\right)^{m}\left(a^{\dagger} - b^{\dagger}\right)^{n}|0\rangle\langle 0|a^{m}b^{n}, \tag{S43}$$

so the induced Hamiltonian becomes

$$H_{\flat}' = g(a^{\dagger}b + ab^{\dagger}). \tag{S44}$$

Note that this induced Hamiltonian is indeed Hermitian. This shows that, in addition to the finite dimensional cases, the Hamiltonian of infinite dimension can also be Hermitized via the vielbein formalism.