# Continuous Dissipative Phase Transitions without Symmetry Breaking

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The paradigm of second-order phase transitions (PTs) induced by spontaneous symmetry breaking (SSB) in thermal and quantum systems is a pillar of modern physics that has been fruitfully applied to out-of-equilibrium open quantum systems. Dissipative phase transitions (DPTs) of second order are often connected with SSB, in close analogy with well-known thermal second-order PTs in closed quantum and classical systems. That is, a second-order DPT should disappear by preventing the occurrence of SSB. Here, we prove this statement to be wrong, showing that, surprisingly, SSB is not a necessary condition for the occurrence of second-order DPTs in *out-of-equilibrium open quantum systems*. We analytically prove this result using the Liouvillian theory of dissipative phase transitions, and demonstrate this anomalous transition in a paradigmatic laser model, where we can arbitrarily remove SSB while retaining criticality, and on a $Z_2$-symmetric model of a two-photon Kerr resonator. This new type of a phase transition cannot be interpreted as a "semiclassical" bifurcation, because, after the DPT, the system steady state remains unique.

## I. INTRODUCTION

The similarities and differences between quantum (or thermal) phase transitions (PTs) and dissipative phase transitions (DPTs) in open quantum systems are the subject of intense research [1–8]. Criticality and critical phenomena (e.g., hysteresis [9–12] and slowing-down [13–15]) have been predicted, observed, and characterized for first-order DPTs. Central to the characterization of second-order PTs is the role of spontaneous symmetry breaking (SSB): nonanaliticity can occur when a system symmetry is "broken", i.e., the emergence of several steady (or ground) states that are not invariant anymore under the action of a given symmetry group [16–18]. SSB in open systems has been discussed in, e.g., Refs. [19–25]. The relation between criticality, symmetries, and exotic effects have also been discussed for a wide range of models [5, 26–30].

In this article, we analytically prove that *second-order DPTs in open quantum systems can occur with or without symmetry breaking*. Similarly to other examples, where the phases of dissipative systems possess features which have no analogue in closed and thermal systems [19, 31–34], this feature can be explained by the spectral properties of a Liouvillian superoperator (i.e., the generator of the dynamics of an out-of-equilibrium open quantum system) [1, 18, 35].

We demonstrate our results with two examples. First, we consider a lasing model characterized by $U(1)$-SSB with a second-order DPT. By adding dephasing, the $U(1)$ symmetry of the model is maintained, but its phase coherence is destroyed, thus preventing SSB. Yet, a second-order DPT takes place. Second, we consider the $Z_2$ symmetry breaking in parametric down-conversion. In this case, the addition of a parity dissipator enables a DPT without $Z_2$-SSB.

Similar phenomena of continuous PTs without symmetry breaking (whose explanation goes beyond the Landau theory of PTs [17, 36, 37]), can be encountered in closed systems at *equilibrium* [38, 39]. These examples are characterized by a nontrivial topological structure, e.g., topological insulators [40, 41]. As such, our work prompts the question for a generalization of topological PTs in *non-equilibrium*, non-quadratic, and bosonic systems [42–44].

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## II. CRITICALITY OF OPEN QUANTUM SYSTEMS

Under the Born and Markov approximations [45], the reduced density matrix $\hat{\rho}(t)$ of an open quantum system at time $t$ evolves according to a Lindblad master equation ($\hbar = 1$):

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = \mathcal{L}\hat{\rho}(t) = -i\left[\hat{H}, \hat{\rho}(t)\right] + \sum_j \mathcal{D}[\hat{L}_j]\hat{\rho}(t), \tag{1}$$

where $\hat{H}$ is the Hamiltonian describing the coherent part of the system evolution, $\mathcal{L}$ is the Liouvillian superoperator [35, 45–47], and $\mathcal{D}[L_j]$ are the so-called Lindblad dissipators, whose action is

$$\mathcal{D}[\hat{L}_j]\hat{\rho}(t) = \hat{L}_i\hat{\rho}(t)\hat{L}_j^\dagger - \frac{\hat{L}_j^\dagger\hat{L}_j\hat{\rho}(t) + \hat{\rho}(t)\hat{L}_j^\dagger\hat{L}_j}{2}. \tag{2}$$

The operators $\hat{L}_j$ are the jump operators, and they describe how the environment acts on the system inducing loss and gain of particles, energy, and information. The steady state $\hat{\rho}_{\rm ss}$, i.e., the state which does not evolve anymore under the action of the Liouvillian ($\partial_t\hat{\rho}_{\rm ss} = \mathcal{L}\hat{\rho}_{\rm ss} = 0$), is central to a system characterization. We indicate the expectation values of operators at the steady state as $\langle\hat{o}\rangle_{\rm ss} = \mathrm{Tr}[\hat{\rho}_{\rm ss}\hat{o}]$.

A DPT is a discontinuous change in $\hat{\rho}_{\rm ss}$ as a function of a single parameter [1, 18]. For instance, the medium gain rate $A$, defined in Eq. (11), plays this role in the first model considered below, and:

$$\lim_{A\to A_c}\frac{\partial^2}{\partial A^2}\hat{\rho}_{\rm ss}(A) \to \infty, \tag{3}$$

where $A_c$ is the critical point.

This thermodynamic nonanalyticity can be witnessed in finite-size systems, as discussed in Refs. [1, 18] and experimentally demonstrated in Refs. [10, 15, 48]. Criticality is accompanied by the emergence of the so-called *critical slowing down*, i.e., the appearance of infinitely-long timescales in the system dynamics. Diverging timescales can be captured by the Liouvillian spectrum, defined by

$$\mathcal{L}\hat{\rho}_i = \lambda_i\hat{\rho}_i, \tag{4}$$

$\lambda_i$ (the eigenvalues) representing the decay rates and oscillation frequencies, and $\hat{\rho}_i$ (the eigenmatrices) encoding the states explored along the dynamics of $\mathcal{L}$.

## III. DISSIPATIVE PHASE TRANSITIONS WITH OR WITHOUT SPONTANEOUS SYMMETRY BREAKING

Before dealing with a specific model, let us provide a demonstration of this novel type of criticality.

The weak symmetry of a dissipative system can be described by a superoperator $\mathcal{U}$ such that $\mathcal{U} = \hat{J}\cdot\hat{J}^\dagger$, where $\hat{J}^\dagger = \hat{J}^{-1}$ [49]. This is always the case if we assume that $\mathcal{U}$ defines a cyclic group (such as $Z_n$) and therefore $\hat{J}^\dagger\hat{J} = \hat{1}$. Note that this is one of the most common types of symmetries that one encounters in open quantum systems, characterizing, e.g., lasing ($U_1$), parametric downconversion ($Z_2$), and $Z_n$ for a translational invariant lattice with $n$ sites.

The fact that the Liouvillian is symmetric, i.e., $[\mathcal{L}, \mathcal{U}] = 0$ allows partitioning the space in different *symmetry sectors*, i.e., parts of the Liouvillian space which are not connected to other parts (sectors) by the system dynamics [30, 50, 51]. Accordingly, the Liouvillian reads $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \dots$, where $\mathcal{L}_k$ is the evolution operator of the $k$-th symmetry sector. As such, we can relabel the eigenvalues and eigenmatrices as $\lambda_i^{(k)}$ and $\hat{\rho}_i^{(k)}$, respectively, and write

$$\mathcal{L}_k\hat{\rho}_j^{(k)} = \lambda_i^{(k)}\hat{\rho}_j^{(k)}, \quad \text{and} \quad \mathcal{U}\hat{\rho}_j^{(k)} = \hat{J}\hat{\rho}_j^{(k)}\hat{J}^\dagger = u^{(k)}\hat{\rho}_j^{(k)}. \tag{5}$$

We order the eigenvalues of each symmetry sector in such a way that $\left|\mathrm{Re}\left[\lambda_0^{(k)}\right]\right| \leq \left|\mathrm{Re}\left[\lambda_1^{(k)}\right]\right| \leq \left|\mathrm{Re}\left[\lambda_2^{(k)}\right]\right| \leq \dots$. In this regard, $\lambda_0^{(k)}$ represents the slowest-decaying process in each symmetry sector. The steady state must always be such that $\mathcal{U}\hat{\rho}_{\rm ss} = \hat{\rho}_{\rm ss}$ [18]. We call this the symmetry sector for $k = 0$, and therefore $\hat{\rho}_0^{(0)} \propto \hat{\rho}_{\rm ss}$. Thus, each eigenmatrix belonging to the symmetry sector for $k = 0$, i.e., $\hat{\rho}_j^{(0)}$, is such that $\mathcal{U}\hat{\rho}_j^{(0)} = \hat{\rho}_j^{(0)}$.

These considerations on symmetries are valid independently of the presence of a DPT, but they are fundamentally related to SSB. Consider now a Liouvillian that, in a certain region of parameter space, admits a unique eigenvalue $\lambda_0^{(0)} = 0$ and all the other eigenvalues are such that $\lambda_0^{(j)} \neq 0$ (i.e., $\hat{\rho}_{\rm ss} \propto \hat{\rho}_0^{(0)}$, such that $\mathcal{L}\hat{\rho}_{\rm ss} = 0$, is the only eigenmatrix which does not evolve under the Lindblad master equation). Within this formalism, a phase transition with SSB means that in each symmetry sector a zero eigenvalue $\lambda_0^{(k)}$ occurs, i.e.,

$$\mathcal{L}\hat{\rho}_0^{(k)} \neq 0 \text{ if } A < A_c, \quad \text{and} \quad \mathcal{L}\hat{\rho}_0^{(k)} = 0 \text{ if } A > A_c, \tag{6}$$

where the critical point $A_c$ is defined in Eq. (3). Let us now consider a new Liouvillian

$$\mathcal{L}' = \mathcal{L} + \mathcal{D}[\hat{L}]. \tag{7}$$

If $\mathcal{L}$ is a well-defined Liouvillian, so does $\mathcal{L}'$, because adding a dissipator to a Liouvillian keeps $\mathcal{L}'$ a completely positive and trace-preserving (CPTP) map. Note that the following property holds

$$\mathcal{D}[\hat{L}]\hat{\rho}_j^{(k)} = \hat{L}\hat{\rho}_j^{(k)}\hat{L}^\dagger - \frac{\hat{L}^\dagger\hat{L}\hat{\rho}_j^{(k)} + \hat{\rho}_j^{(k)}\hat{L}^\dagger\hat{L}}{2} \neq 0, \quad \text{if and only if} \quad k \neq 0, \tag{8}$$

if

$$\left[\hat{\rho}_j^{(0)}, \hat{L}\right] = \left[\hat{\rho}_j^{(0)}, \hat{L}^\dagger\right] = 0, \text{ and } \left[\hat{\rho}_j^{(k)}, \hat{L}\right] = \left[\hat{\rho}_j^{(k)}, \hat{L}^\dagger\right] \neq 0. \tag{9}$$

All $\hat{\rho}_j^{(0)}$ must remain unchanged, while for $k \neq 0$, the dynamics of $\mathcal{L}$ and of $\mathcal{L}'$ must differ. But since $\lambda_j^{(0)} \leq 0$, and the dynamics must be different, it can only occur that $\lambda_j^{(0)} \neq 0$.

This operator can *always* be found. Indeed, we can choose the symmetry operator $\hat{J}$ as a jump operator because

$$\mathcal{D}[\hat{J}]\hat{\rho}_j^{(k)} = \hat{J}\hat{\rho}_j^{(k)}\hat{J}^\dagger - \frac{\hat{J}^\dagger\hat{J}\hat{\rho}_j^{(k)} + \hat{\rho}_j^{(k)}\hat{J}^\dagger\hat{J}}{2} = \mathcal{U}\hat{\rho}_j^{(k)} - \hat{\rho}_j^{(k)} = \left(u^{(k)} - 1\right)\hat{\rho}_j^{(k)} = 0, \quad \text{if and only if} \quad k = 0, \tag{10}$$

according to Eq. (5), and given $\hat{J}^\dagger\hat{J} = \hat{1}$. We conclude that *one can arbitrarily remove SSB from any second-order DPT*. This does not mean that $\hat{J}$ is the only operator which allows removing the SSB while keeping a second-order DPT. It can also be done, e.g, by a generator of a symmetry group $\hat{J}$, provided that $\hat{J}$ is continuous, as we show below on the example of a $U(1)$ model.

A remark is in order. If we were to include an additional term in the Hermitian Hamiltonian, not a Lindbladian dissipator, in Eq. (1), the above reasoning would remain the same. Indeed, by considering $\hat{H}' = \hat{H} + \hat{L}$ (where now $\hat{L}$ needs to be Hermitian), the commutator $[\hat{\rho}_j^{(0)}, \hat{L}] = 0$ ensures that the dynamics is again unchanged for the sector $k = 0$.

## IV. MODEL I: A $U(1)$-SYMMETRIC LASER MODEL

Let us provide an example of a DPT where the SSB can be arbitrarily removed. Consider a laser-like $U(1)$ model with $\hat{H} = \omega\hat{a}^\dagger\hat{a}$ and jump operators

$$\hat{L}_1 = \frac{\hat{a}^\dagger(2A - B\hat{a}\hat{a}^\dagger)}{2\sqrt{A}}, \;\; \hat{L}_2 = \sqrt{\frac{3B}{4}}\,\hat{a}\hat{a}^\dagger, \;\; \hat{L}_3 = \sqrt{\Gamma}\hat{a}, \tag{11}$$

where $\hat{a}$ ($\hat{a}^\dagger$) is the bosonic annihilation (creation) operator, $\hat{L}_1$ describes the laser gain, $\hat{L}_2$ captures the field dephasing, and $\hat{L}_3$ represents the particle loss. The jump operators are characterized by the rates: $A$ for the medium gain, $B$ for the gain saturation, $\Gamma$ for the dissipation (the inverse of the photon lifetime) Changing to the frame rotating at the frequency $\omega$, we can set $\hat{H} = 0$ [30]. This model is the celebrated Scully-Lamb laser master equation in the so-called weak-gain saturation regime [52–55], valid if $A = \mathcal{O}(\Gamma)$ and $B\left\langle\hat{a}\hat{a}^\dagger\right\rangle \ll 2A$. The limits of the validity of this approximation for the system dynamics are detailed in, e.g., Refs. [56, 57].

The model is characterized by a $U(1)$ *weak symmetry* [58, 59], which is represented by the symmetry operator $\hat{J} = \exp\left(i\phi\hat{a}^\dagger\hat{a}\right)$. Indeed, the transformation $\hat{a} \to \hat{J}\hat{a}\hat{J}^\dagger \to \hat{a}e^{i\phi}$ leaves the equation of motion unchanged, but $\left\langle\hat{a}^\dagger\hat{a}(t)\right\rangle$ is *not* conserved. Thus, $\langle\hat{a}\rangle_{\rm ss} = 0$ holds for any finite-size system. A SSB in the thermodynamic limit is,

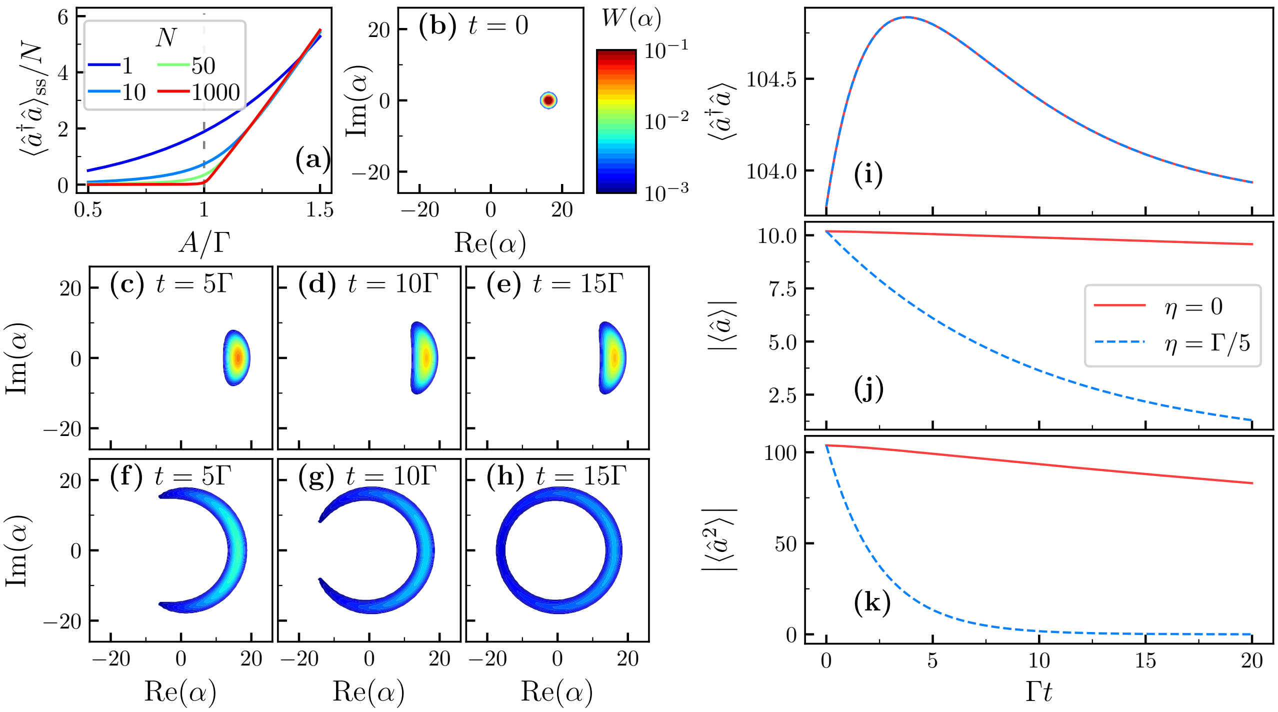


FIG. 1. Dissipative phase transition with or without the $U(1)$ spontaneous symmetry breaking. (a) Rescaled number of photons $\langle \hat{a}^\dagger \hat{a}\rangle_{ss}/N$ versus incoherent drive strength $A/\Gamma$ for various $N$. The solid curves are the results of the quantum simulations, and are independent of $\eta$. (b) Wigner function of a state initialized in the coherent state $|\alpha = \sqrt{\langle \hat{a}^\dagger \hat{a}\rangle_{ss}}\rangle$ for $A = 1.25\Gamma$ and $N = 50$ evolving with: (c-e) $\eta = 0$; (f-h) $\eta = \Gamma/5$. If $\eta = 0$, the dissipative phase transition coincides with a spontaneous symmetry breaking (shown by the long-time coherence). If $\eta \neq 0$, a dissipative phase transition occurs because $\langle \hat{a}^\dagger \hat{a}\rangle_{ss}/N$ is discontinuous, but the system has no spontaneous symmetry breaking since it rapidly loses coherence. The time evolution of the operators $\langle \hat{a}^\dagger \hat{a}\rangle$, $|\langle \hat{a}\rangle|$, and $|\langle \hat{a}^2\rangle|$ for $\eta = 0$ (solid red curve) and $\eta = \Gamma/5$ is shown in (h-k), respectively. Parameters: $B/\Gamma = 10^{-1}/N$ and $\omega/\Gamma = 0$ (i.e., the frame rotates at $\omega$).

thus, signalled by $\langle \hat{a}\rangle (t \to \infty) \neq 0$, which follows from symmetry considerations [60]. Therefore, in finite-size systems, the $U(1)$ SSB is signalled by $\langle \hat{a}\rangle (t) \neq 0$ for increasingly long time, as we increase $N$. To better grasp the meaning of this symmetry, let us express the eigenmatrices $\hat{\rho}_i$ of the Liouvillian and the action of the symmetry operator in Eq. (5) in the number (Fock) basis:

$$\hat{\rho}_j^{(k)} = \sum_{m,n} c_{m,n}^{(k)} |m\rangle\langle n| \quad \text{and} \quad \mathcal{U}\hat{\rho}_j^{(k)} = \sum_{m,n} c_{m,n}^{(k)} e^{-i\phi \hat{a}^\dagger \hat{a}} |m\rangle\langle n| e^{i\phi \hat{a}^\dagger \hat{a}} = \sum_{m,n} c_{m,n}^{(k)} e^{-i\phi(m-n)} |m\rangle\langle n| = u_i^{(k)} \hat{\rho}_i^{(k)}. \tag{12}$$

We conclude that $\exp[-i\phi(m-n)]$ must be a constant and, therefore, any eigenmatrix $\hat{\rho}_i$ in Eq. (12) must obey

$$\hat{\rho}_i^{(k)} = \sum_m c_m^{(k)} |m\rangle\langle m-k| \,, \tag{13}$$

for some constant integer $k \in \mathbb{Z}$. In other words, $\hat{\rho}_i^{(k)}$ must be an operator containing elements only on one diagonal, and different symmetry sectors occupy different upper and lower diagonals.

### A. Second-order DPT with the $U(1)$ SSB

PTs and nonanaliticity can only emerge in the thermodynamic limit. One can exploit the infinite dimension of the bosonic Hilbert space to observe a nonanalytical change in the steady state, as discussed, in, e.g. Refs. [18, 20, 30, 61–63]. To do so, we consider a rescaling parameter $N$, so that the size of the Hilbert space increases, but a meaningful observable, such as the rescaled photon number, merge far from the critical point:

$$\{A, B, \Gamma\} \to \{A, B/N, \Gamma\}. \tag{14}$$

In a laser model [52], $N$ represents an increasing number of injected three-level atoms in the lase cavity, but each with a weaker light-matter coupling. We also refer to Ref. [57] for a more detailed discussion of the system thermodynamic limit and on the nature of the DPT in the Scully-Lamb laser model.

To numerically simulate the results of this model, we introduce a cutoff $C$ in the Hilbert space, i.e., we assume that $\langle m|\hat{\rho}(t)|n\rangle = 0$ if $m > C$ or $n > C$. We then verify the convergence with the cutoff, i.e., we check that by increasing $C$ the values do not change (within a numerical precision). By decreasing the nonlinearity (i.e., increasing $N$), $C$ increases.

In Fig. 1(a), we plot the rescaled photon number $\left\langle \hat{a}^\dagger \hat{a}\right\rangle_{\rm ss}/N$, obtained by numerically solving $\mathcal{L}\hat{\rho}_{\rm ss} = 0$ (solid curves), and the results of the semiclassical approximation (dashed lines) for different values of $N$. As one can see, the photon number is continuous, but there is an emerging "elbow" signalling the occurrence of a second-order DPT.

The breaking of the $U(1)$ symmetry means the retaining of coherence for infinitely long time. Although this occurs only in the thermodynamic limit, we can show the presence of the critical slowing down for finite-size systems. In Figs. 1(b)-1(e), we show the Wigner function $W(\alpha)$ for an initially coherent state when $\eta = 0$ in the "broken symmetry region" (i.e., for $A > A_c = \Gamma$), where $W(\alpha) = 2\,{\rm Tr}\left[\hat{D}_\alpha \exp\left(i\pi \hat{a}^\dagger \hat{a}\right) \hat{D}_\alpha^\dagger \hat{\rho}(t)\right]/\pi$ and $\hat{D}_\alpha = \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right)$ is the displacement operator [64]. As time passes, the system retains its coherence $\langle \hat{a}(t)\rangle \neq 0$, signalling that, even for this finite-size system, a critical timescale associated with SSB has emerged.

To better quantify the meaning of SSB and the role of the $U(1)$ symmetry, we note that

$$ {\rm Tr}\left[\hat{a}^n \hat{\rho}_j^{(k)}\right] = {\rm Tr}\left[\mathcal{U}^\dagger \mathcal{U} \hat{a}^n \hat{\rho}_j^{(k)}\right] = {\rm Tr}\left[\left(e^{+i\phi \hat{a}^\dagger \hat{a}} \hat{a}^n e^{-i\phi \hat{a}^\dagger \hat{a}}\right)\left(e^{-i\phi \hat{a}^\dagger \hat{a}} \hat{\rho}_j^{(k)} e^{i\phi \hat{a}^\dagger \hat{a}}\right)\right] = e^{-i\phi(k-n)}\left\langle \hat{a}^n \hat{\rho}_j^{(k)}\right\rangle. \tag{15} $$

We, thus, conclude that the critical slowing down, associated with SSB in the $k$th symmetry sector, is witnessed by $\left\langle \hat{a}^k\right\rangle$. Thus, in Figs. 1(i-k) we plot with red solid curves the dynamics of $\left\langle \hat{a}^\dagger \hat{a}(t)\right\rangle$, $|\left\langle \hat{a}(t)\right\rangle|$, and $|\left\langle \hat{a}^2(t)\right\rangle|$ (associated with the sectors $k = 0$, $\pm 1$, $\pm 2$), respectively. The initial state is the same coherent state as in Figs. 1(b). While the photon number rapidly converges to its steady-state value, we see again that the coherences are preserved for very long-times, confirming that indeed a $U(1)$ SSB is taking place, and there is a critical slowing down in each symmetry sector.

### B. Removing the $U(1)$ SSB

According to our proof, we should be able to remove the SSB while retaining criticality. To do that, we notice that a jump operator of the form $\hat{L} = \sqrt{\eta/4}\hat{a}\hat{a}^\dagger$, where $\eta$ represents an additional dephasing rate, satisfies Eq. (9) given the structure of $\hat{\rho}_j^{(k)}$ [c.f. Eq. (13)].

First, we verified that the photon number is identical to the one for which $\eta = 0$ [the results are within a floating-point precision in Fig. 1(a)]. Again, the photon number becomes sharper and sharper with increasing $N$, and the results coincide for all tested values of $\eta$, meaning that a second-order DPT is taking place.

Figs. 1(f)-1(h) show that an initially coherent state, as in the case $\eta \neq 0$, rapidly looses its coherence on a timescale of $\eta/2$. That is, the $U(1)$ SSB does not take place. Indeed, $\hat{\rho}(t) \simeq \hat{\rho}_{\rm ss}$ in Fig. 1(h), indicating that an initial state rapidly reaches its steady state, proving the absence of any residual or hidden SSB, which would anyhow lead to a critical slowing down.

To better quantify how each symmetry sector is affected, we compare $\left\langle \hat{a}^\dagger \hat{a}(t)\right\rangle$, $|\left\langle \hat{a}(t)\right\rangle|$, and $|\left\langle \hat{a}^2(t)\right\rangle|$ in Figs. 1(i-k) for $\eta = 0$ (red solid curves) and $\eta \neq 0$ (blue dashed curves). Wile the sector for $k = 0$ is unaffected (as demonstrated by the photon number). Dynamics in other symmetry sectors is much faster, and $|\left\langle \hat{a}^n(t)\right\rangle|$ rapidly reaches zero.

We have, thus, demonstrated the existence of a second-order DPT without the $U(1)$ SSB and the possibility to arbitrary remove the SSB retaining a critical behavior. Importantly, the semiclassical analysis [60] fails to capture this kind of criticality (see also Ref. [57] for details), which highlights the necessity to use the Liouvillian formalism in describing dissipative critical phenomena.

## V. MODEL II: A $Z_2$-SYMMETRIC KERR RESONATOR

To further illustrate the validity of our results, here we consider the second-order DPT of a two-photon Kerr resonator, studied in, e.g., [18, 20, 21, 26]. The Hamiltonian in the frame rotating at the pump frequency reads

$$ \hat{H} = -\Delta \hat{a}^\dagger \hat{a} + i\frac{G}{2}\left[\left(\hat{a}^\dagger\right)^2 - \hat{a}^2\right] + \frac{U}{2}\left(\hat{a}^\dagger\right)^2 \hat{a}^2, \tag{16} $$

where $\Delta$ is the cavity-to-pump detuning, $G$ is the two-photon drive intensity, and $U$ is the Kerr nonlinear interaction.

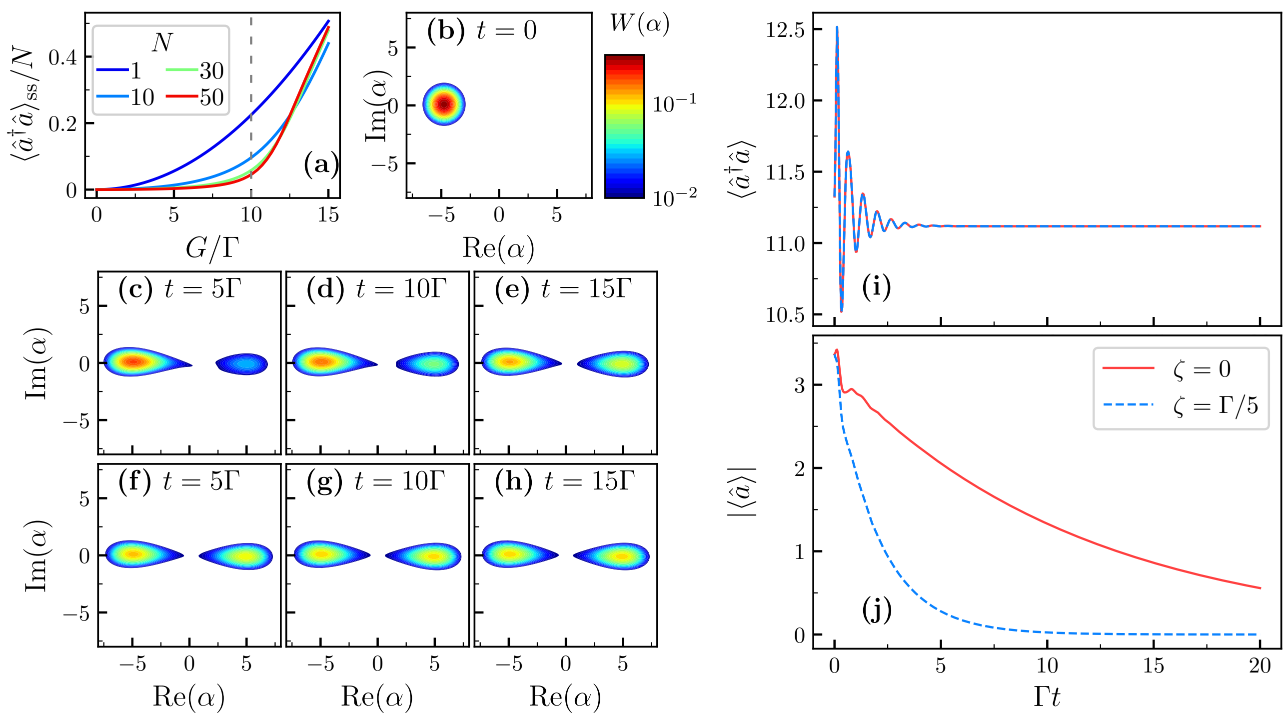


FIG. 2. Dissipative phase transition with or without the $Z_2$ spontaneous symmetry breaking. (a) Rescaled number of photons $\langle \hat{a}^\dagger \hat{a} \rangle_{ss}/N$ versus incoherent drive strength $A/\Gamma$ for various $N$. The solid curves are the results of the quantum simulations, and are independent of $\zeta$. (b) Wigner function of a state initialized in the coherent state $|\alpha = \sqrt{\langle \hat{a}^\dagger \hat{a} \rangle_{ss}}\rangle$ for $G = 12.5\Gamma$ and $N = 50$ evolving with: (c-e) $\zeta = 0$; (f-h) $\zeta = \Gamma/5$. If $\zeta = 0$, the dissipative phase transition coincides with a spontaneous symmetry breaking (shown by the long-time coherence). If $\eta \neq 0$, a dissipative phase transition occurs because $\langle \hat{a}^\dagger \hat{a} \rangle_{ss}/N$ is discontinuous, but the system has no spontaneous symmetry breaking since it rapidly loses coherence. The time evolution of the operators $\langle \hat{a}^\dagger \hat{a} \rangle$ and $|\langle \hat{a} \rangle|$ for $\zeta = 0$ (solid red curve) and $\zeta = \Gamma/5$ are shown in (i-j), respectively. Parameters: $\Delta/\Gamma = 10$ and $U/\Gamma = 10$.

Photons continuously escape the Kerr resonator, and the system is described by the Lindblad master equation for the system density matrix $\hat{\rho}(t)$ [65]

$$\frac{\partial}{\partial t}\hat{\rho}(t) = -i[\hat{H}, \hat{\rho}(t)] + \Gamma \mathcal{D}[\hat{a}]\hat{\rho}(t). \tag{17}$$

This system is characterized by a $Z_2$ weak symmetry, meaning that $\hat{a} \to \hat{J}\hat{a}\hat{J}^\dagger = -\hat{a}$ leaves the equation of motion unchanged, where $\hat{J} = \exp\left(i\pi \hat{a}^\dagger \hat{a}\right)$, but parity is not a conserved quantity [59]. With a reasoning similar to that of Eq. (12), we can demonstrate that there are two symmetry sectors, $k = 0$ and $k = 1$, such that

$$\mathcal{U}\hat{\rho}_i^{(k)} = e^{ik\pi}\hat{\rho}_i^{(k)}, \text{ and } \hat{\rho}_i^{(k)} = \sum_{m,n} c^{(k)}_{2m,\,2n}|2m\rangle\langle 2n+k| + c^{(k)}_{2m+1,\,2n+1}|2m+1\rangle\langle 2m+1+k|\,. \tag{18}$$

We conclude that $\hat{\rho}_i^{(0)}$ contains only the even-even and odd-odd states, while $\hat{\rho}_i^{(1)}$ couples the even-odd and odd-even states.

This time, the rescaling parameter $N$ acts as $\{\Delta, U, G, \Gamma\} \to \{\Delta, U/N, G, \Gamma\}$ [20, 66, 67].

Similarly to the previous case, we observe the emergence of a second-order DPT [Fig. 2(a)]. This is accompanied by a critical slowing down signaling a SSB, as it can be argued from the evolution of a system initialized in a coherent state. In particular, both the Wigner functions in Figs. 2(b-e) and the evolution of $\langle \hat{a}^\dagger \hat{a} \rangle$ and $|\langle \hat{a} \rangle|$ [solid lines in Figs. 2(i-j)] show the presence of a slow-time scale in the $k = 1$ symmetry sector.

To remove the SSB and keep the second-order DPT, this time we consider an additional jump operator of the form $\hat{L} = \sqrt{\zeta}\hat{J} = \sqrt{\zeta}\exp\left(i\pi\hat{a}^\dagger\hat{a}\right)$. Again, there is no difference with the case for $\zeta = 0$ in the symmetry sector for $k = 0$, as it can be argued by the fact that the photon number in the steady state is unchanged [Fig. 2(a)] as well as the time dynamics of $\langle \hat{a}^\dagger \hat{a} \rangle$ [Fig. 2(i)]. However, the presence of $\zeta$ significantly changes the sector $k = 1$, as it can be seen by analyzing the Wigner function in Figs. 2(f-h) and the time evolution of $|\langle \hat{a} \rangle|$ in Fig. 2(j).

We confirm again our predictions, and we show that, by adding an appropriate dissipator, we can remove the SSB also in the case of this $Z_2$ SSB.

## VI. CONCLUSIONS

In this article, we have proved that continuous DPTs can occur with and without an SSB. In particular, we derived analytical conditions to remove SSB from any second-order DPT. As examples, we have analyzed a paradigmatic non-equilibrium lasing system and a model characterized by a discrete $Z_2$ symmetry. In both cases, our analytical predictions are confirmed by the numerical simulations, demonstrating how tremendously multiform and various are the non-equilibrium states, their dynamics, and their phase transitions.

The presented DPTs without SSB are, to our knowledge, a novel phenomenon, where dissipation plays a fundamental role. This phenomenon opens questions concerning the mechanism of criticality in open quantum systems. Our predictions can be experimentally tested with, e.g., superconducting circuits, where physically engineered dissipation can be realized with state-of-the-art techniques. From a fundamental point of view, the presence of second-order DPTs, where degeneracy can be removed, is intriguing because it represents a shift from the Landau theory of phase transitions. Revealing a link, if any, with extensions of topological theories for DPTs is one of our future objectives [43, 44]. Even though the models considered here were characterized by a single order parameter, our results are valid, in general, for any multi-order parameter systems with any global symmetry, e.g., open and/or dissipative extensions of thermal $\mathbb{Z}_2$-symmetric systems with light-matter interactions [68, 69].

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