Supplementary Materials for: Restoring Adiabatic State Transfer in Time-Modulated Non-Hermitian Systems

by I. I. Arkhipov *et al.*

I. FIDELITY BETWEEN TIME-EVOLVING RIGHT EIGENSTATES AND INITIAL LEFT EIGENSTATES

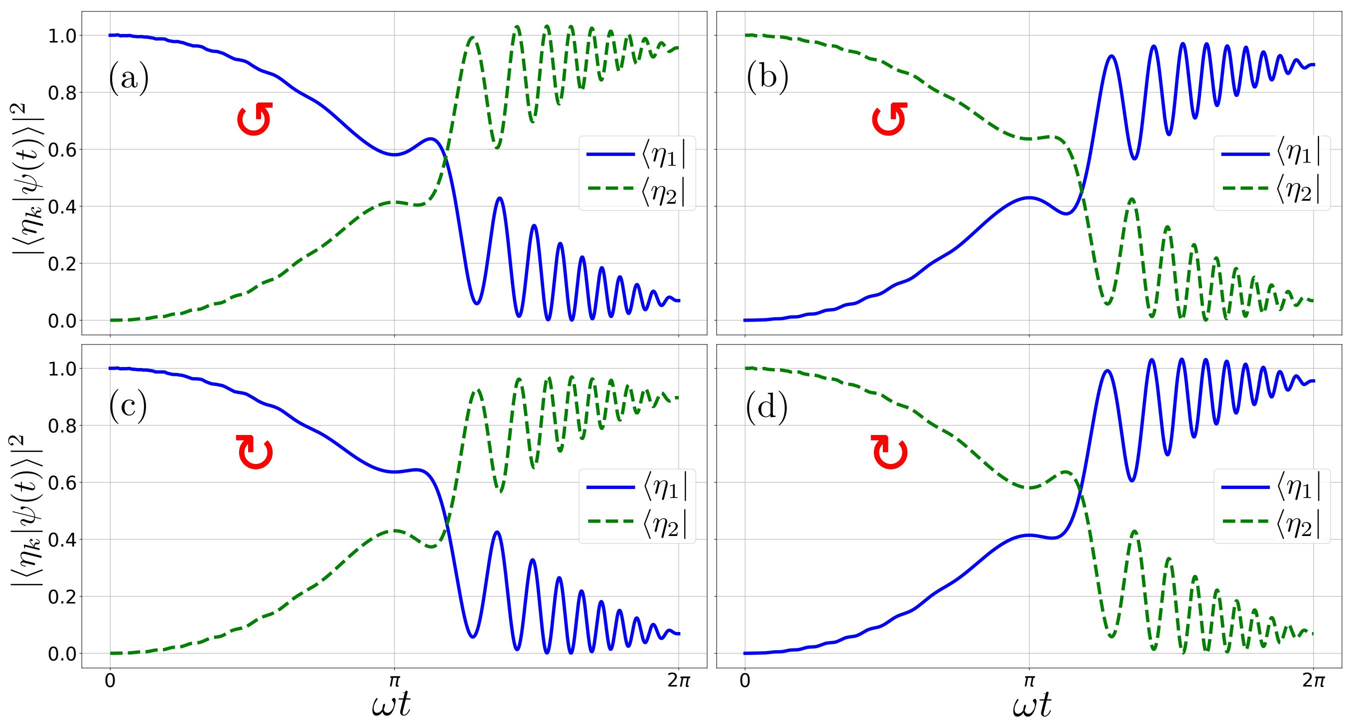

FIG. S1. Fidelity $F_{jk} = |\langle\eta_j|\psi(t)_k\rangle|^2$ between the time-evolving right eigenstate $|\psi_k(t)\rangle$ and initial left eigenvector $|\eta_j\rangle$ at time $t = 0$ of the NHH in Eq. (3) while encircling the EP. Panels (a)-(b) show counterclockwise winding and (c)-(d) show clockwise winding, similar to that in Fig. 4 in the main text. The other parameters are: $r = 0.5$, $\omega = \pi/50$, and $\phi_0 = \pi$, according to Eq. (6) in the main text.

In Fig. 4 in the main text we have shown the plots for calculated fidelitites between time-evolving right eigenstates $|\psi(t)\rangle$ and instantaneous left eigenstates $|\eta\rangle$ of the Hamiltonian in Eq. (3). To further corroborate on the observed adiabatic state transfer, here in Fig. S1, we plot the fidelities between $|\psi(t)\rangle$ and the initial left eigenmodes $|\eta(t=0)\rangle$ instead, which exhibit the same final switched states. As can be seen on panels (a) and (d) in Fig. S1, the fidelity can slightly exceed the value 1. This seeming anomaly stems from the fact that the biorthogonality condition $\langle\eta_i|\psi_j(t)\rangle = \delta_{ij}$ is only ensured for the initial time $t = 0$.

II. EFFECT OF INCLUDING A σ_y TERM IN THE NON-HERMITIAN HAMILTONIAN IN EQ. (1)

Here we discuss the effect of the inclusion of the σ_y term in the NHH H in Eq. (1) in the main text. We show that our main conclusions, presented in the main text, remain unaffected by this incorporation, provided that it is handled appropriately.

Indeed, let us consider a NHH of the form

$$H_{\rm tot} = d_x\sigma_x + d_y\sigma_y + d_z\sigma_z, \tag{S1}$$

where $\vec{d} = [d_x, d_y, d_z]$ is a complex-valued vector. By rotating this NHH by an angle θ via the following unitary

$$U_\theta = \cos\frac{\theta}{2} I_2 + i \sin\frac{\theta}{2}\sigma_x, \tag{S2}$$

the modified NHH $H_\theta = U_\theta^\dagger H_{\rm tot} U_\theta$ reads

$$H_\theta = d_x\sigma_x + (d_y\cos\theta - d_z\sin\theta)\sigma_y + (d_y\sin\theta + d_z\cos\theta)\sigma_z. \tag{S3}$$

By setting $\tan\theta = d_y/d_z$ in Eq. (S3), the σ_y term is effectively eliminated. We also require that $\theta = \text{const}$, in order to ensure that the state dynamics, governed by the Schrödinger equation, does not change under rotation. Obviously, this can be achieved by modulating the d_y coefficient in the same manner as d_z, up to the scaling factor $\tan\theta$. The latter allows to reduce the system dynamics to that described by the NHH in Eq. (1), namely

$$H_\theta = d_x\sigma_x + \frac{d_z}{\cos\theta}\sigma_z, \tag{S4}$$

by identifying d_x and $d_z/\cos\theta$ with the corresponding coefficients of the NHH in Eq. (1). This implies that the eigenstates of the NHH in Eq. (S1) $|\psi_{\rm tot}\rangle$ can be simply related to the eigenstates $|\psi\rangle$ of the NHH H in Eq. (1) as $|\psi_{\rm tot}\rangle = U|\psi\rangle$, which guarantees the preservation of the adiabatic state transfer for $H_{\rm tot}$.

III. STABILITY ANALYSIS OF THE OBSERVED ADIABATIC MODE SWITCHING

In this section, we analyze the impact of NHH perturbations on adiabatic state transfer when dynamically encircling an EP, as discussed in the main text.

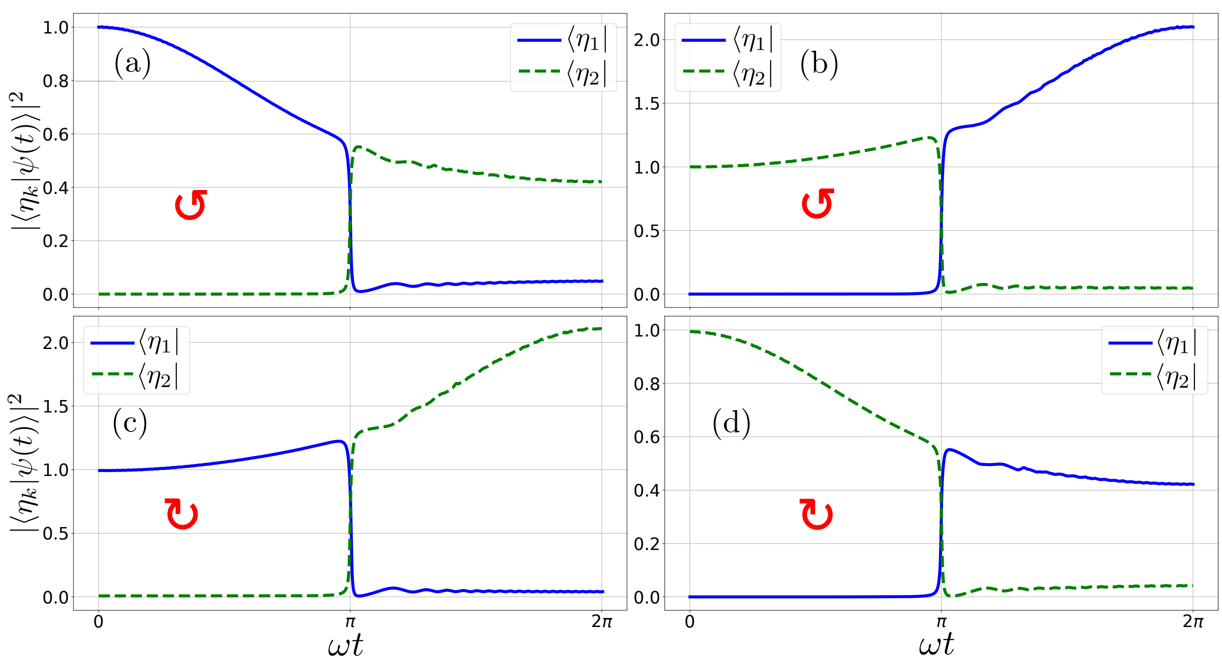

FIG. S2. Fidelity $F_{jk} = |\langle\eta_j|\psi(t)_k\rangle|^2$ between the time-evolving right eigenstate $|\psi_k(t)\rangle$ and the static left eigenvector $|\eta_j\rangle$ of the NHH in Eq. (S5) while encircling the EP. Panels (a)-(b) show counterclockwise winding and (c)-(d) show clockwise winding, similar to that in Fig. 4 in the main text. The other parameters are: $\delta = 0.01$, $r = 0.5$, $\omega = \pi/100$, and $\phi_0 = \pi$.

Depending on the Hamiltonian perturbations, the state evolution can change drastically. This occurs because perturbations can render the NHH eigenvalues complex-valued, which may result in the occurrence of non-adiabatic transitions (NATs) in the system dynamics.

As it was already pointed in the main text, a simple shift of the NHH eigenvalues on some constant imaginary value, i.e., $H' \to H + i\delta I$, where I is the identity matrix, and $\delta \in \mathbb{R}$, does not have any effect on the adiabatic state

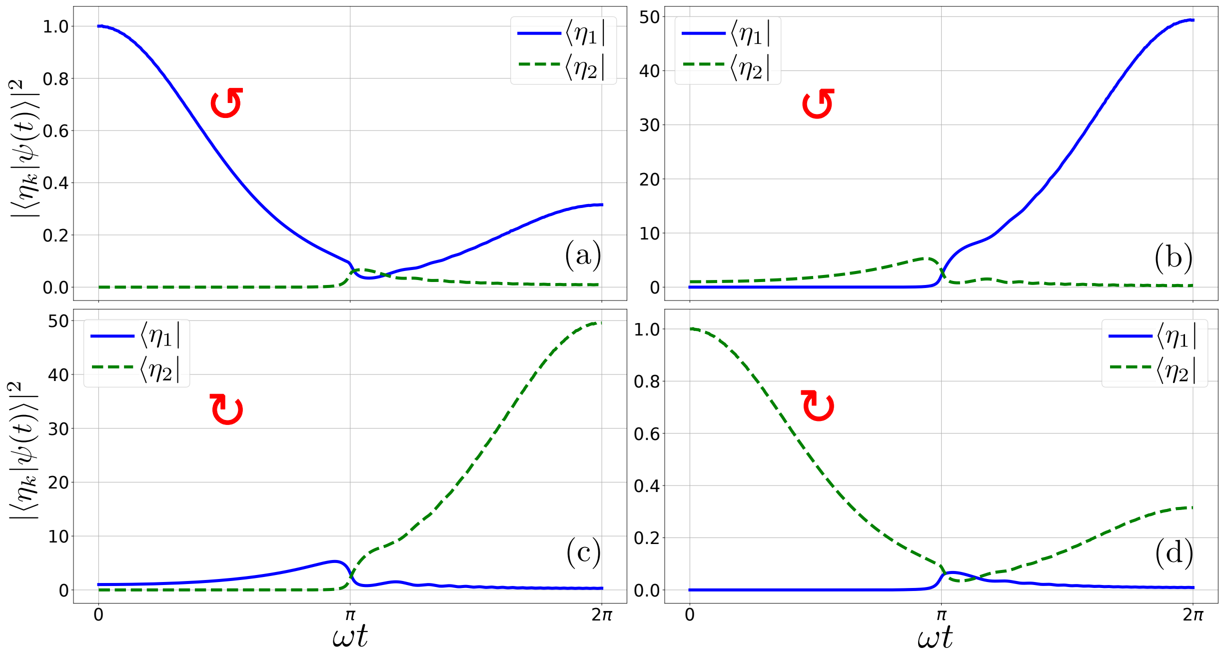

FIG. S3. Fidelity $F_{jk} = |\langle\eta_j|\psi(t)_k\rangle|^2$ between the time-evolving right eigenstate $|\psi_k(t)\rangle$ and the static left eigenvector $|\eta_j\rangle$ of the NHH in Eq. (S5) while encircling the EP. Panels (a)-(b) show counterclockwise winding and (c)-(d) show clockwise winding, similar to that in Fig. 4 in the main text. The other parameters are: $\delta = 0.05$, $r = 0.5$, $\omega = \pi/100$, and $\phi_0 = \pi$.

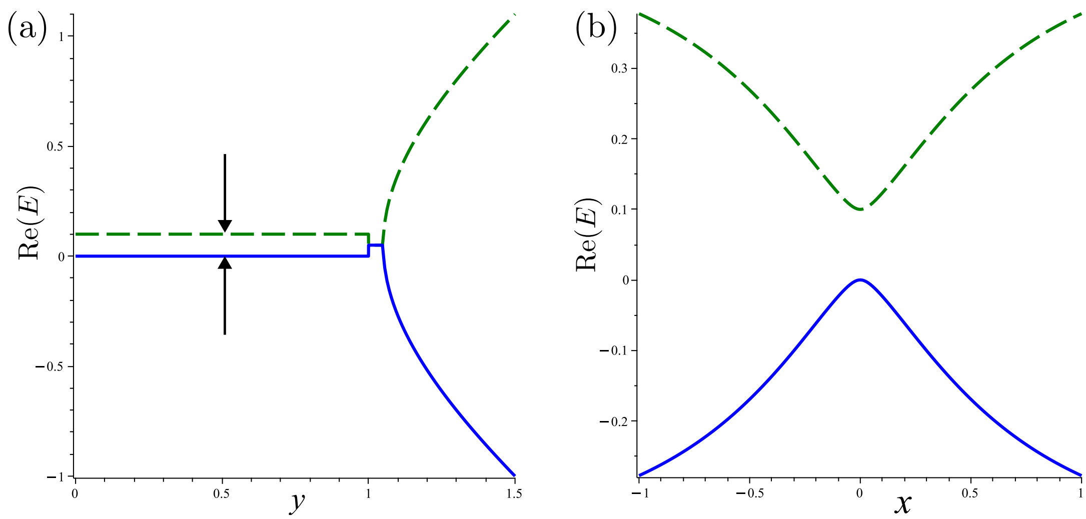

FIG. S4. Real part of the spectrum of the NHH H' in Eq. (S5). (a) The cross-section of the spectrum for $x = 0$ when $\delta = 0.1$. The perturbation thus leads to the appearance of the gap in the real-valued energy. The state trajectory in Eq. (6) in the main text traverses this gap at $y = 0.5$ (also shown by the vertical black arrows). (b) The cross-section of the spectrum for $y = 0.5$. Other system parameters are the same as in Fig. 4. For small values of $\delta \ll 1$, the adiabatic state transfer can still be realized via Landau-Zener-Stueckelberg-Majorana transitions [1], when the time period is finite. The latter ensures that the evolving state 'diabatically jumps' between two energy manifolds at the gap $x = 0$.

transfer. Indeed, for the NAT to occur, the imaginary parts of the two eigenvalues must substantially differ. The latter ensures that while dynamically encircling an EP, the system may exhibit a non-adiabatic transition, when the system prefers the eigenstate which belongs to the energy manifold with the least loss (imaginary part) [2].

On the other hand, by perturbing the NHH in Eq. (3) as

$$H' = H + \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}, \tag{S5}$$

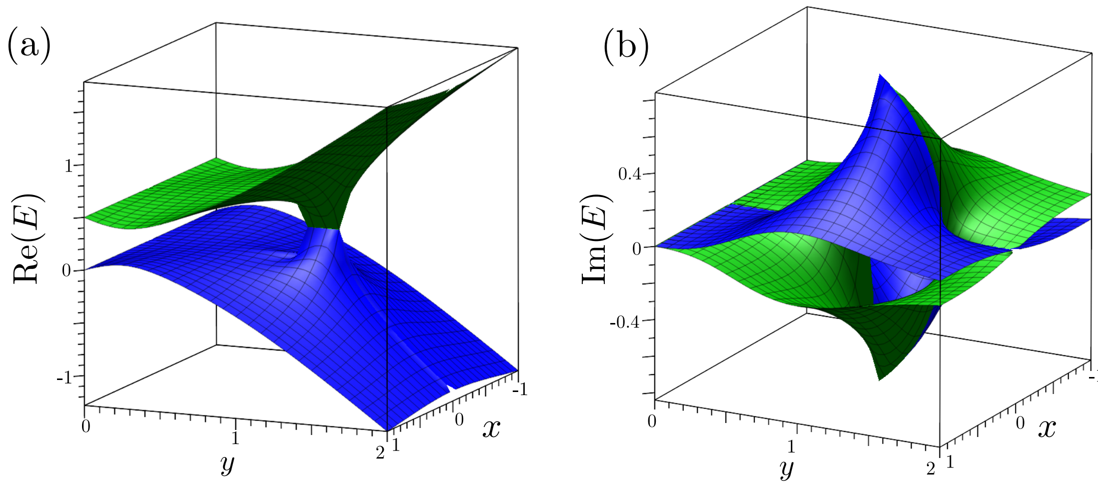

FIG. S5. Real (a) and imaginary (b) parts of the spectrum of the NHH H' in Eq. (S5) in the whole (x, y) space at larger values of perturbations (here $\delta = 0.5$). At larger δ, the chiral dynamics solely stems from the NATs.

the eigenenergies attain the form

$$E'_{1,2} = \frac{1}{2}\left(\delta \mp \sqrt{\delta^2 + 4\alpha^2 + 4\alpha\delta\cos\phi}\right). \tag{S6}$$

According to Eq. (S6), the difference in imaginary parts of the eigenvalues $\mathrm{Im}(E'_2 - E'_1)$ increases with $|\delta|$, which can eventually lead to the emergence of NATs in the system dynamics, and, in particular, to a chiral mode behaviour. We illustrate the perturbation effects on the fidelity, which is calculated identically to that shown in Fig. 4 in the main text, in Fig. S2 and Fig. S3 for $\delta = 0.01$ and $\delta = 0.05$, respectively. The rest of the system parameters are the same as in Fig. 4.

As can be seen in Fig. S2, the adiabatic state transfer persists for very small nonzero $\delta = 0.01$. However, the evolving states start acquiring an exponential factor, because of the breakdown of the pseudo-Hermiticity of the non-Hermitian Hamiltonian, due to the appearance of the imaginary eigenvalues of the NHH H'. This exponential factor results either in the damping or amplification of the evolving state [panels (a), (d) and panels (b), (c) in Fig. S2, respectively].

For nonzero δ, the diabolic degeneracy is always lifted by the value δ when $\alpha = x = 0$ (see Fig. S4). This means that the adiabatic state transfer between the two energy manifolds can still be realized via Landau-Zener-Stueckelberg-Majorana transitions for finite time periods [1]. However, at larger values of perturbation ($\delta \geq 0.05$), the adiabaticity breaks down, leading to the occurrence of NATs in the system, and, consequently, to a chiral mode behaviour, as shown in Fig. S3. For small $\delta \ll 1$, one can expect the occurrence of both the Landau-Zener-Stueckelberg-Majorana transitions and NATs in the system dynamics, as corroborated by panels (a) and (d) in Fig. S3. Thus, the chirality here can stem from the intricate interplay of both transitions.

By further increasing δ, the difference in both the real and imaginary eigenenergies increases (see Fig. S5). This implies that by winding around the EP on the real energy surface, assuming no NATs, would confine the evolving eigenstate to the corresponding manifolds, given the negligible probability of Landau-Zener-Stueckelberg-Majorana transitions. However, since $\mathrm{Im}(E'_2 - E'_1)$ can be significantly greater than zero along the trajectory, the exhibited chirality originates solely from the NATs, where again the final state would depend only on the winding direction. Indeed, suppose that one starts in the $|\psi_1\rangle$ state corresponding to the blue energy surface on Fig. S5(a), similar to that in Fig. 3(a) in the main text. Now, by encircling the EP in the clockwise direction, the state remains on the given manifold as long the imaginary part of E'_1 is larger than that of E'_2 [green blue surface in Fig. S5(b)]. In the time span $T/2 < t < T$, the $\mathrm{Im}(E'_2) > \mathrm{Im}(E'_1)$ [corresponding to the sector $(x < 0)$ in Fig. S5(b)], as a result, a NAT would occur, leading to the transition of the evolving state $|\psi(t)\rangle \to |\psi_2(t)\rangle$, which now belong to the energy surface E'_2 (green surface), thus resulting in the effective mode switch $|\psi_1\rangle \to |\psi_2\rangle$ at $t = T$.

The same line of argumentation applies to the initial state $|\psi_2\rangle$ when winding in the same clockwise direction. In that case, for $t < T/2$, corresponding to the sector $(x > 0)$, the evolving state would exhibit a NAT $|\psi(t)\rangle \to |\psi_1(t)\rangle$, since $\mathrm{Im}(E_1') > \mathrm{Im}(E_2')$ in that time region. However, for times $t > T/2$, the resulting state is again $|\psi_2\rangle$ after completing the full cycle, due to the occurrence of the second NAT, as was described above. In other words, regardless of the initial state, the clockwise encirclement results in the $|\psi_2\rangle$ state. And vice versa, for the counterclockwise winding, the final state is always $|\psi_2\rangle$.

[1] O. V. Ivakhnenko, S. N. Shevchenko, and F. Nori, "Nonadiabatic Landau–Zener–Stückelberg–Majorana transitions, dynamics, and interference," Phys. Rep. **995**, 1–89 (2023).

[2] J. Doppler, A. A. Mailybaev, J. Böhm, U. Kuhl, A. Girschik, F. Libisch, T. J. Milburn, P. Rabl, N. Moiseyev, and S. Rotter, "Dynamically encircling an exceptional point for asymmetric mode switching," Nature (London) **537**, 76 (2016).