Restoring Adiabatic State Transfer in Time-Modulated Non-Hermitian Systems

Ievgen I. Arkhipov,[1,*] Fabrizio Minganti,[2,3] Adam Miranowicz,[4,5,6] Şahin K. Özdemir,[7] and Franco Nori[4,5,8]

[1]*Joint Laboratory of Optics of Palacký University and Institute of Physics of CAS, Faculty of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic*

[2]*Institute of Physics, Ecole Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland*

[3]*Center for Quantum Science and Engineering, Ecole Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland*

[4]*Theoretical Quantum Physics Laboratory, Cluster for Pioneering Research, RIKEN, Wako-shi, Saitama 351-0198, Japan*

[5]*Quantum Information Physics Theory Research Team, Quantum Computing Center, RIKEN, Wakoshi, Saitama, 351-0198, Japan*

[6]*Institute of Spintronics and Quantum Information, Faculty of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland*

[7]*Department of Engineering Science and Mechanics, and Materials Research Institute (MRI), The Pennsylvania State University, University Park, Pennsylvania 16802, USA*

[8]*Physics Department, The University of Michigan, Ann Arbor, Michigan 48109-1040, USA*

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Non-Hermitian systems have attracted much interest in recent decades, driven partly by the existence of exotic spectral singularities, known as exceptional points (EPs), where the dimensionality of the system evolution operator is reduced. Among various intriguing applications, the discovery of EPs has suggested the potential for implementing a symmetric mode switch, when encircling them in a system parameter space. However, subsequent theoretical and experimental works have revealed that *dynamical* encirclement of EPs invariably results in asymmetric mode conversion; namely, the mode switching depends only on the winding direction but not on the initial state. This chirality arises from the failure of adiabaticity due to the complex spectrum of non-Hermitian systems. Although the chirality revealed has undoubtedly made a significant impact in the field, a realization of the originally sought symmetric adiabatic passage in non-Hermitian systems with EPs has since been elusive. In this work, we bridge this gap and theoretically demonstrate that adiabaticity, and therefore a symmetric state transfer, is achievable when dynamically winding around an EP. This becomes feasible by specifically choosing a trajectory in the system parameter space along which the corresponding evolution operator attains a real spectrum. Our findings, thus, offer a promise for advancing various wave manipulation protocols in both quantum and classical domains.

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Introduction—Non-Hermitian, i.e., nonconservative or open, systems are characterized by a complex-valued spectrum. Because of this complexity, such systems can exhibit spectral singularities, known as exceptional points (EPs), where both eigenvalues and eigenvectors of a corresponding evolution operator coalesce [1]. The existence of EPs introduces a variety of rich and intriguing phenomena not encountered in conservative, i.e., Hermitian, systems [2–4].

In the quantum realm, the non-Hermitian evolution operator can be represented by either a non-Hermitian Hamiltonian (NHH) or a Liouvillian, depending on whether the portrayal of the system dynamics excludes or includes the effects of quantum jumps, respectively [5]. Consequently, in the description of nonconservative (semi) classical systems or open quantum systems upon post-selection, exclusive reliance on the NHH formalism is usually sufficient [6].

Since the discovery of EPs, it has been anticipated that EPs can be potentially exploited for adiabatic state transfer, due to the Riemann topology of the system spectrum induced by these singularities [7–11]. That is, by winding around an EP, one can symmetrically switch between system eigenstates thanks to the presence of a branch cut between two energy Riemann surfaces [12]. Adiabatic evolution implies that this state transfer only depends on an initial state, not on a winding direction. This observation was also confirmed experimentally when realizing *stationary*, i.e., time-independent NHHs [13–17].

However, subsequent theoretical [18–20] and later experimental [21–28] works have demonstrated that for time-dependent NHHs the adiabaticity assumption breaks down due to the imaginary part of the system spectrum. Namely, when one dynamically encircles an EP, the system inevitably experiences nonadiabatic transitions (NATs), which lead to a state-flip asymmetry. This

*Contact author: ievgen.arkhipov@upol.cz

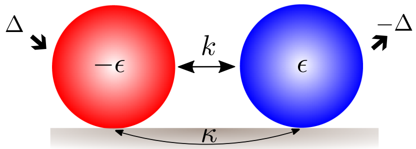

FIG. 1. Schematic representation of an open system described by a non-Hermitian Hamiltonian H in Eq. (1). The system consists of two cavities, detuned in frequency $\pm\epsilon$, and coupled coherently with interaction strength k, and dissipatively with strength κ. Both cavities can be amplified with gain rate Δ or experience losses with rate $-\Delta$.

appearing chirality ensures that only the orbiting direction determines the final state. A recent work [29] showed that the state flip symmetry can be recovered in dissipative systems by exploiting the spectral topology of hybrid diabolic-exceptional points [30], but only through non-adiabatic transformations in *multimode* systems.

While the chiral mode behavior revealed in time-modulated non-Hermitian systems has, undoubtedly, led to important advancements in the field, still, achieving the originally sought symmetric adiabatic passage in such systems has remained elusive. In this work, we bridge this gap and demonstrate that one can restore adiabatic symmetric state transfer in open systems while dynamically orbiting around an EP. This becomes feasible thanks to a specific choice of the encircling trajectory in a system parameter space. The protocol proposed here relies on a proper mapping of the system parameter space of a given NHH onto a certain submanifold, where the NHH becomes pseudo-Hermitian, i.e., a Hamiltonian with real eigenvalues. Compared to systems with NATs, which are usually associated with instabilities, our protocol exploits the system real spectrum and therefore can provide greater system control and robustness. The latter property is especially crucial in the quantum domain. We illustrate our findings with the simplest example of a dissipative two-level system. Our results thus hold promise for advancing light manipulation protocol in both quantum and classical domains.

Model—We consider a two-level NHH

$$H = (k + i\kappa)\sigma_x + (\epsilon - i\Delta)\sigma_z, \tag{1}$$

where $\sigma_{x,z}$ are Pauli matrices, and $k, \kappa, \epsilon, \Delta \in \mathbb{R}$. This NHH can describe either a classical two-level system or a quantum one, subjected to postselection in some global decaying reference frame [6]. As shown in Fig. 1, a possible realization of such a system is two coupled dissipative cavities (in the mode representation), where Δ ($-\Delta$) denotes the resonator gain (loss) rate and ϵ is the frequency detuning of the resonators. The parameters k and κ account for coherent and incoherent, i.e., dissipative, mode coupling strengths, respectively [30–32]. This NHH determines the state evolution via the Schrödinger equation $i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle$.

The Hamiltonian in Eq. (1) has complex eigenvalues $E_{\mp} = \mp\sqrt{[-\Delta + k - i(\epsilon - \kappa)][\Delta + k + i(\kappa + \epsilon)]}$ and admits EPs in its parameter space defined by equations $|\Delta_{\rm EP}| = |k|$ and $|\epsilon_{\rm EP}| = |\kappa|$. When $\kappa = 0$, the NHH in Eq. (1) reduces to the paradigmatic classical two-mode model used for the demonstration of the chiral mode behavior [21,33]. Indeed, while encircling the EP, the imaginary part of the eigenenergies $E_{\pm}$ plays a fundamental role in determining which state "survives" at the end of the winding protocol, due to induced NATs, resulting thus in a chiral state transfer. Evidently, the same conclusion holds true when dynamically encircling other EPs in the system's complex energy space.

Mapping a non-Hermitian Hamiltonian onto a pseudo-Hermitian one—In light of the emergent chiral mode behavior in non-Hermitian systems with EPs, we are motivated by the following question: Can one dynamically encircle an EP without inducing NATs during the system evolution? In other words, can one restore adiabatic state transfer in a time-modulated non-Hermitian system? Below we show that the answer is affirmative. This becomes possible provided that, along the orbiting trajectory in a system parameter space, a given NHH acquires a pseudo-Hermitian form, i.e., it attains a real spectrum.

This pseudo-Hermitian transformation of the NHH in Eq. (1) can be achieved, in particular, with the help of a certain function $f\colon \vec{r} = (x, y) \to (\Delta, \epsilon, k, \kappa)$, which maps a two-dimensional (2D) real space (x, y), called a chart, onto a manifold in the 4D parameter space of the NHH. The minimal dimension of the chart is 2D since for finding an EP one needs only two parameters [34]. The sought manifold can be parametrized as follows:

$$\begin{aligned} \Delta &= \alpha \sinh\phi_i \sin\phi_r, & \epsilon &= \alpha\cosh\phi_i\cos\phi_r, \\ k &= \alpha\cosh\phi_i \sin\phi_r, & \kappa &= \alpha\sinh\phi_i\cos\phi_r, \end{aligned} \tag{2}$$

where $\phi = \phi_r + i\phi_i = \arctan[(x + iy)^{-1}] \in \mathbb{C}$, and $\alpha = x\sinh\phi_i/\sin\phi_r \in \mathbb{R}$. This manifold describes a 4D hyperboloid with a nonconstant curvature α determined by the relation $\epsilon^2 + k^2 - \Delta^2 - \kappa^2 = \alpha^2$. High-dimensional hyperboloid space parametrizations find applications in various fields of mathematics and physics. A notable example is the anti–de Sitter spacetime with a constant curvature, establishing the anti–de Sitter and conformal field theory (AdS/CFT) correspondence, which links quantum gravity to quantum field theories [35].

With the help of Eq. (2), the Hamiltonian H in Eq. (1) and its eigenenergies E acquire the form

$$H = \alpha\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}, \qquad E_{1,2} = \mp\alpha. \tag{3}$$

The embedding in Eq. (2) (an injective continuous map) guarantees that the NHH has real eigenvalues $\pm\alpha$ on the whole (x, y) chart. In other words, this parametrization is the one that ensures that the corresponding 4D system parameter hyperboloid results in real-valued eigenvalues. The corresponding right eigenvectors of H are

$$|\psi_1\rangle \equiv \left[-\sin\frac{\phi}{2}, \cos\frac{\phi}{2}\right]^T, \qquad |\psi_2\rangle \equiv \left[\cos\frac{\phi}{2}, \sin\frac{\phi}{2}\right]^T, \tag{4}$$

where T stands for transpose. Together with the left eigenvectors [36]

$$|\eta_1\rangle \equiv \left[-\sin\frac{\phi^*}{2}, \cos\frac{\phi^*}{2}\right]^T, \quad |\eta_2\rangle \equiv \left[\cos\frac{\phi^*}{2}, \sin\frac{\phi^*}{2}\right]^T, \tag{5}$$

they form the biorthogonal basis, i.e., $\langle\eta_j|\psi_k\rangle = \delta_{jk}$. The introduction of the left eigenvectors, i.e., dual vector space, is necessary since the right eigenvectors alone are nonorthogonal [37].

Explicitly, the acquired pseudo-Hermiticity of the NHH in Eq. (3) on a given 4D hyperboloid means that there is a Hermitian operator ξ, such that $H^\dagger = \xi^{-1}H\xi$ [39], and this symmetry is exact on the whole parametrized 4D space. The operator ξ does not necessarily express this symmetry globally, i.e., independently of the system parameters, but can exhibit it locally, as can be expected from the chosen parametrization. Obviously, one can take $\xi = MM^\dagger$, where M is the diagonalizing matrix of H, whose columns are formed by the right eigenvector $|\psi_{1,2}\rangle$ in Eq. (4) [40].

The eigenenergy manifold of the NHH is characterized by the Riemann topology, where two real-valued energy surfaces wrapped around two EPs on the (x, y) plane at $\vec{r}_{\rm EP} = (0, \pm 1)$ (see Fig. 2). Accordingly, in the 4D hyperboloid parameter space, these EPs are defined at $\epsilon_{\rm EP} = \mp 1$, $\Delta_{\rm EP} = k_{\rm EP} = 0$, and $\kappa_{\rm EP} = 1$ [43]. Moreover, the branch cut corresponds to the finite diabolic zero-energy line, with these two EPs on its ends (Fig. 2). The right eigenvectors become equivalent at the EPs, up to a certain global phase, namely, $|\psi_1\rangle_{\rm EP} \equiv \exp(\pm i\pi/2)|\psi_2\rangle_{\rm EP}$, respectively.

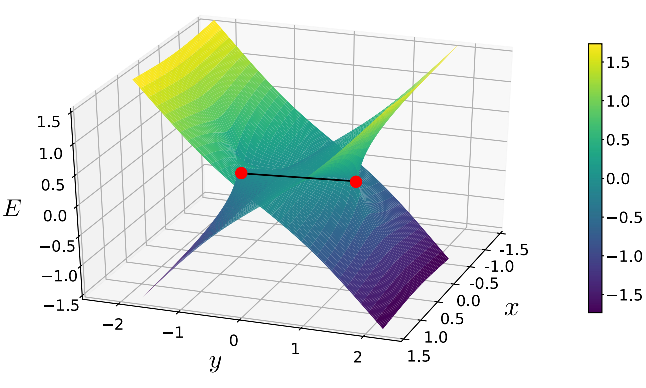

FIG. 2. Eigenenergy spectrum E of the pseudo-Hermitian Hamiltonian H given in Eq. (3). The spectrum consists of energy Riemann surfaces wrapped around two EPs (red circles). The spectrum is purely real on the (x, y) plane, and, therefore, on the corresponding 4D hyperboloid in the system parameter space described in Eq. (2).

Adiabatic state transfer while dynamically encircling an EP—Here we demonstrate that the dynamics governed by the time-dependent NHH in Eq. (3) is free from NATs, implying that the states can adiabatically evolve along the orbits on the parametrized space while dynamically encircling the EP. Let us first define the system time-evolution trajectory as

$$\begin{aligned} x(t) &= r\sin(\omega t + \phi_0), \\ y(t) &= 1 - r\cos(\omega t + \phi_0), \end{aligned} \tag{6}$$

where $r, \omega, \phi_0 \in \mathbb{R}$ are constants, and the time t is presented in arbitrary units. Correspondingly, we change $H(t)$ in Eq. (3). The path in Eq. (6) describes a circle with a radius r on the plane (x, y), whose center is at the EP, $\vec{r}_{\rm EP} = (0, 1)$ [see Figs. 4(a) and 4(b)]. The starting point corresponds to the phase $\phi_0 = \pi$, where the two energy levels $E_{1,2}$ are maximally separated. For the angular frequency $\omega > 0$ ($\omega < 0$), the orbiting trajectory goes counterclockwise (clockwise). This circle trajectory on the chart corresponds to a loop on the surface of the 4D hyperboloid due to the embedding nature of the map f (see also Fig. 3).

We initialize the system in one of the right eigenstates in Eq. (4), namely, $|\psi(t = 0)\rangle = |\psi_k\rangle$, and then we find the evolving state $|\psi(t)\rangle$ by numerically integrating the

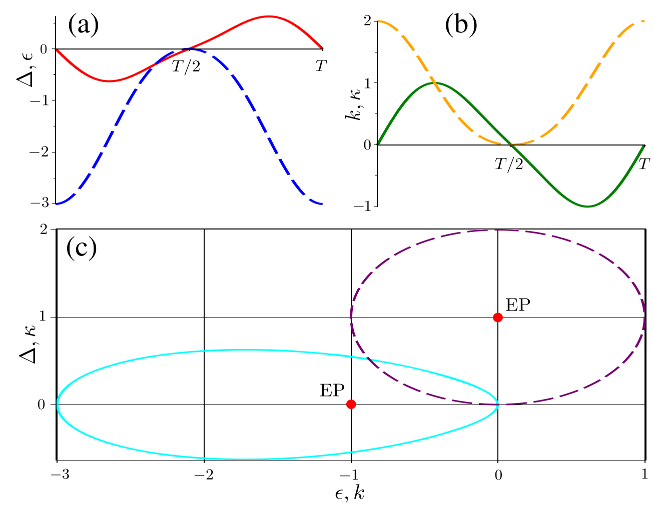

FIG. 3. Time-modulated system parameters in Eq. (2) when winding in the counterclockwise direction in the chart (x, y), according to Eq. (6). (a) Gain-loss rates Δ (red solid curve), and frequency detuning ϵ (blue dashed curve). (b) Coherent k (green solid curve) and incoherent κ (orange dashed curve) mode-coupling strengths, respectively. (c) A projection of the corresponding loop on a surface of the 4D hyperboloid, in the system parameter space, onto the subspaces (ϵ, Δ) (cyan solid curve) and (k, κ) (purple dashed curve), respectively. Red points denote the same EP in both subspaces, which correspond to that in Fig. 2 for $(x = 0, y = 1)$. In all panels, the winding radius is set at $r = 0.5$ and the angular frequency is $\omega = 2\pi$.

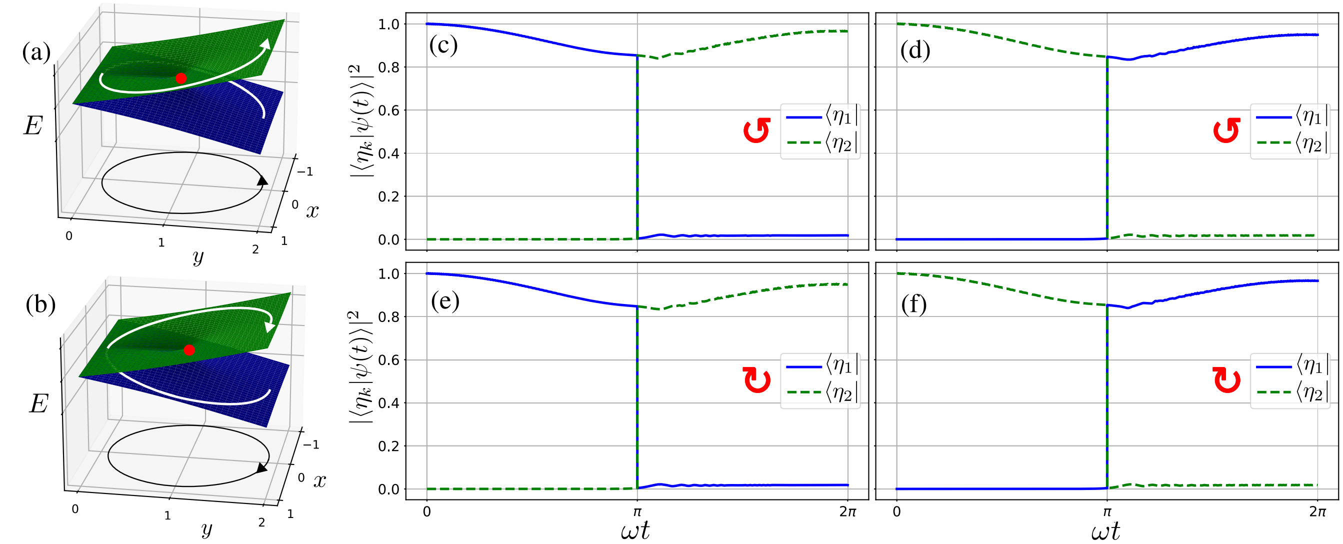

FIG. 4. Adiabatic state transfer while dynamically encircling around an EP in the (x, y) plane. Schematic representation of the clockwise (a) and counterclockwise (b) winding direction. In (a),(b) the initial state is $|\psi_1\rangle$, corresponding to the E_1 energy surface (blue surface). (c)–(f) Fidelity $F_{jk} = |\langle\eta_j|\psi(t)_k\rangle|^2$ between the time-evolving right eigenstate $|\psi_k(t)\rangle$ and static left eigenvector $|\eta_j\rangle$ of the NHH in Eq. (3) while encircling the EP. Panels (c)–(d) Counterclockwise winding and (e)–(f) Clockwise winding. The other parameters are $r = 0.5$, $\omega = \pi/100$, and $\phi_0 = \pi$, according to Eq. (6). The right eigenstates are exchanged after the complete dynamical cycle regardless of the winding direction. The state dynamics exhibits a purely adiabatic character with no NATs; thus, enabling one to implement a symmetric state switch.

Schrödinger equation. To track the state dynamics, we calculate the fidelity $F_{jk} = |\langle\eta_j|\psi(t)\rangle|^2$, i.e., the overlap of $|\psi(t)\rangle$ with the instantaneous left eigenvector $|\eta_j\rangle$ of the NHH [44]. Instantaneous in the sense that for each time step we independently calculate the left eigenvectors of the NHH in Eq. (5) by substituting the parameters obtained by $[x(t), y(t)]$ in Eq. (5) [45]. The results of these calculations for different winding directions and initialized states are shown in Fig. 4. For completeness, we also calculate the fidelity between evolving right eigenstates and left eigenmodes at $t = 0$ in the Supplemental Material [46].

As one can see in Fig. 4, the eigenstates $|\psi_1\rangle \to |\psi_2\rangle$ and $|\psi_2\rangle \to |\psi_1\rangle$ are exchanged after one period $T = 2\pi/\omega$, regardless of the encircling direction, exhibiting thus the adiabatic character of the state evolution. For instance, the system initialized in the state $|\psi_1\rangle$ [blue lower energy surface in Figs. 4(a) and 4(b)], continuously evolves until the branch cut, corresponding to the half of the period $T/2$ [see Figs. 4(c) and 4(e)]. After that, the evolving state starts moving on the upper green energy surface, corresponding to the eigenstate with energy E_2. Thus, after completing the full cycle, independently on the encircling path, the initial eigenstate $|\psi_1\rangle$ is switched to the eigenstate $|\psi_2\rangle$, and vice versa [see Figs. 4(d) and 4(f)].

Such an adiabatic behavior is in striking contrast to the previously studied non-Hermitian systems with EPs, where the presence of complex Riemann topology always leads to the nonadiabatic jumps during the state evolution. Here, on the other hand, thanks to the purely real spectrum of the NHH along the chosen trajectory, the presence of NATs is eliminated, allowing us to restore the adiabatic nature of the non-Hermitian state dynamics.

Discussion—The results shown above remain unaffected by incorporating an additional σ_y term in the NHH in Eq. (1), provided that it is handled appropriately [46]. Also, the mode switching protocol presented does not depend on the encircling radius r in Eq. (6), if the trajectories for different r remain equivalent in the sense that they encircle the same number of EP(s) [47].

When considering the stability of the observed adiabatic dynamics upon a perturbation of the NHH in Eq. (3) as $H' = H + H_\delta$, its behavior can vary drastically depending on the nature of the disturbance. Evidently, a perturbation that breaks the pseudo-Hermiticity of the NHH in Eq. (3) may substantially affect the state evolution. If the perturbation just shifts the NHH spectrum by an imaginary constant ($H_\delta = -i\gamma I_2$, where I_2 is the identity matrix), the adiabaticity is preserved and no NATs are induced in the system [48]. This property can be utilized, e.g., for implementing a symmetric mode converter in purely dissipative quantum systems, i.e., with no gain [50,51]. However, by perturbing the system, e.g., as $H_\delta = \mathrm{diag}[\delta, 0]$, where diag stands for a diagonal matrix, the induced difference $\nu \sim \mathrm{Im}\left[\sqrt{\delta^2 + 4\alpha^2 + 4\alpha\delta\cos\phi}\right]$ in the imaginary parts for the two eigenvalues of H' becomes larger for larger $|\delta|$. The latter can eventually lead to the emergence of NATs in the system dynamics and, therefore, to a chiral state transfer for $\nu \gg 0$ [52] (see details in the

Supplemental Material [46]). Furthermore, this perturbation-induced chirality, if controlled, can also enable one to switch between symmetric and asymmetric regimes on demand.

Conclusions—We have demonstrated that time-modulated non-Hermitian systems can exhibit pure adiabatic dynamics while dynamically encircling EP in their parameter space. This is in striking contrast to previous works, where the system complex spectrum always leads to NATs during state evolution, and therefore to asymmetric state transfer. Remarkably, the adiabaticity can be eventually restored by properly mapping the system parameter space onto a certain manifold, over which a given NHH becomes pseudo-Hermitian with a real spectrum. In particular, this procedure allows to realize a long-sought symmetric mode converter, where system eigenstates are always dynamically swapped regardless of the EP winding direction.

Evidently, the presented results also echo the adiabatic rapid passage (ARP) protocol in Hermitian systems, where a symmetric state switch is realized by adiabatically driving a system along closed loops through diabolic points (DPs) in a system parameter space [54,55]. In that respect, our findings can be treated as a non-Hermitian extension of the ARP. Indeed, the ARP protocol contains crossings of DPs, our switch protocol involves crossings of the diabolic lines while encircling the EPs. Though our protocol is purely adiabatic, whereas the ARP includes a diabatic (i.e., rapid) stage.

Because of the absence of NATs, this observed mode switching mechanism, which exploits energy Riemann topology, among others, can also hold promise for advancing the field of holonomic computations [56–58]. Our findings, thus, open new avenues for the development of novel light manipulation protocols in both classical and quantum photonics.

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[37] The nonorthogonality of the right eigenvectors can be interpreted as the "nonflatness" of the NHH Hilbert space. Such a curved space must be endowed with a metric tensor g, and the left eigenvectors allow us to determine one; namely, $g \equiv \sum |\eta_i\rangle\langle\eta_i|$ [38,39]. The metric g facilitates the proper definition of the inner product on a nonflat space between two arbitrary (right) vectors $|\psi_{j,k}\rangle$ as $\langle\psi_i|\psi_j\rangle_{\mathrm{flat}} \to \langle\psi_i|g|\psi_j\rangle_{\mathrm{curved}}$, which ensures that the vector's norm and, therefore, a state probability are preserved in time [38].
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[47] The final state can be chosen to coincide with the initial state if the trajectory winds, e.g., around two EPs (see Fig. 2). In other words, for the fixed final state, the chosen paths must be equivalent, i.e., belong to the same homotopy class.
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