Quasi-probability distribution $Q(\alpha, \alpha^*)$ versus phase distribution $P(\theta)$ in a description of superpositions of coherent states

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The generation of discrete superpositions of coherent states in the anharmonic oscillator model is discussed from the point of view of their quasi-probability distribution $Q(\alpha, \alpha^*)$ and phase probability distribution $P(\theta)$. It is shown that for the superposition with well-distinguishable states both distributions show the same rotational symmetry. The maximum number of well-distinguishable states is estimated. The two functions are illustrated graphically to show explicitly their symmetry and the influence of the interference terms. The similarity between the Q function integrated over the amplitude and the phase distribution $P(\theta)$ is shown to exist for the anharmonic oscillator states.

1. INTRODUCTION

Generalized coherent states that differ from coherent states by the presence of extra phase factors appearing in the decomposition of such states into a superposition of the Fock states were introduced by Titulaer and Glauber[1] and discussed by Stoler.[2] Bialynicka-Birula[3] showed that, under appropriate periodic conditions, generalized coherent states become discrete superpositions of coherent states. Yurke and Stoler[4] and Tombesi and Mecozzi[5] discussed the possibility of generating quantum-mechanical superpositions of macroscopically distinguishable states in the course of evolution of the anharmonic oscillator. The anharmonic oscillator model was earlier used by Tanaś[6] to show a high degree of squeezing in the model for large numbers of photons. The two-mode version of the model was used by Tanaś and Kielich[7] to describe nonlinear propagation of light in a Kerr medium, predicting a high degree of what was called self-squeezing of strong light. A comparison of quantum and classical Liouville dynamics of the anharmonic oscillator was made by Milburn[8] and Milburn and Holmes.[9] Kitagawa and Yamamoto[10] used the model in their discussion of the number-phase minimum-uncertainty state that can be obtained in a nonlinear Mach–Zehnder interferometer with a Kerr medium. They introduced the term crescent squeezing for squeezing obtained in the model, in contrast to elliptic squeezing of an ordinary squeezed state. The terms crescent and elliptic stem from the shapes of the corresponding contours of the quasi-probability distribution $Q(\alpha, \alpha^*, t)$. The anharmonic oscillator model was also discussed, by Peřinova and Lukš,[11] from the point of view of photon statistics and squeezing. Miranowicz *et al.*[12] recently showed that superpositions with not only even but also odd numbers of components can be obtained. They also showed that the maximum number of well-distinguishable states is proportional to the field amplitude $|\alpha_0|$ and that the quasi-probability distribution $Q(\alpha, \alpha^*, t)$ indicates such superpositions in a spectacular fashion. Recently Gantsog and Tanaś[13] discussed phase properties of self-squeezed states generated by the anharmonic oscillator, using the Hermitian phase formalism introduced recently by Pegg and Barnett.[14–16] Gantsog and Tanaś showed that in cases in which a discrete superposition of coherent states appears, the phase probability distribution splits into separate peaks. If this distribution is plotted in a polar coordinate system, the rotational symmetry of the phase distribution is clearly visible, and it can be compared with the symmetry of the quasi-probability distribution function $Q(\alpha, \alpha^*, t)$. So the phase distribution function $P(\theta)$ can be considered as an alternative with respect to the $Q(\alpha, \alpha^*, t)$ representation of the quantum state of the field, which clearly indicates superpositions of coherent states.

In this paper we compare two descriptions of the field that are a discrete superposition of coherent states. Analytical formulas for the two distributions are obtained for the states generated in the anharmonic oscillator model. It is shown that the two functions show the same rotational symmetry when the superposition states are well separated. The graphs of the two functions show explicitly their rotational symmetry when the states are well separated, as well as the growing contribution of the interference terms, which breaks down the symmetry for a large number of components in the superposition. The maximum number of well-distinguishable states is estimated. It is shown that, in the case of anharmonic oscillator states, there is a close similarity between the Q function integrated over the amplitude and the phase probability distribution $P(\theta)$.

2. ANHARMONIC OSCILLATOR EVOLUTION AND DISCRETE SUPERPOSITIONS OF COHERENT STATES

The model is defined by the Hamiltonian

$$H = \hbar\omega a^{+}a + H_I, \tag{1}$$

with

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$$H_I = \frac{1}{2}\hbar\kappa a^{+2}a^2 = \frac{1}{2}\hbar\kappa\hat{n}(\hat{n}-1), \quad (2)$$

where a and a^+ are the annihilation and creation operators of the field mode and κ is the coupling constant, which is real and can be related to the nonlinear susceptibility $\chi^{(3)}$ of the medium when the model is used to describe nonlinear propagation of light in a Kerr medium.[6,7] We assume that the medium is lossless.

Since the number of photons, $\hat{n} = a^+a$, is a constant of motion, the Heisenberg equations of motion for the field operators can be solved exactly, which allows for derivation of exact analytical solutions for the field variances and prediction of a high degree of squeezing[6,7] in the model.

For the purposes of this paper we need the state evolution rather than the operator evolution. Since the interaction Hamiltonian (2) commutes with the free part of the Hamiltonian (1), the free evolution of the state can be factored out (we will drop it altogether below), and the state evolution of the system is described by the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}U(t) = H_I U(t), \quad (3)$$

where $U(t)$ is the time evolution operator and H_I is the interaction Hamiltonian (2). In the propagation problem, when the light propagates in a Kerr medium, one can make the substitution $t = -z/v$ to describe the spatial evolution of the field instead of the time evolution. After this substitution the solution to Eq. (3) is given by[10]

$$U(\tau) = \exp[i(\tau/2)\hat{n}(\hat{n}-1)], \quad (4)$$

where

$$\tau = \kappa z/v \quad (5)$$

is the dimensionless length of the nonlinear medium (or time in the time domain). If the state of the incoming beam is a coherent state $|\alpha_0\rangle$, the resulting state of the outgoing beam is given by

$$|\psi(\tau)\rangle = U(\tau)|\alpha_0\rangle = \exp\left(-\frac{|\alpha_0|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha_0^{\,n}}{\sqrt{n!}}\exp\left[i\frac{\tau}{2}n(n-1)\right]|n\rangle. \quad (6)$$

The state (6) has an additional phase factor with respect to the coherent state $|\alpha_0\rangle$, and because of the quadratic dependence on n this extra phase cannot be simply added to the phase of the coherent state. So the state (6) differs essentially from the coherent state $|\alpha_0\rangle$. It is known[6–11] that such states lead to squeezing, and they have been referred to as self-squeezed.[6,7] On introducing the notation

$$\alpha_0 = |\alpha_0|\exp(i\varphi_0), \quad (7)$$

$$b_n = \exp(-|\alpha_0|^2/2)(|\alpha_0|^n/\sqrt{n!}), \quad (8)$$

we can rewrite Eq. (6) as

$$|\psi(\tau)\rangle = \sum_{n=0}^{\infty} b_n \exp\left\{i\left[n\varphi_0 + \frac{\tau}{2}n(n-1)\right]\right\}|n\rangle. \quad (9)$$

On account of the presence of the extra phase factor, the state (9) belongs to a class of generalized coherent states.[1,2] It was shown by Bialynicka-Birula[3] that under periodic conditions the generalized coherent states, like Eq. (9), become a discrete superposition of N coherent states and that the superposition coefficients can be found by the solution of a system of N algebraic equations. Such a system of equations was solved for several N values by Miranowicz *et al.*,[12] who obtained analytical formulas for the superposition with both even and odd numbers of components. Averbukh and Perelman[17] recently considered the problem of evolution of wave packets formed by highly excited states of quantum systems, showing the possibility of fractional revivals of the initial wave packet. Their calculations effectively lead to an anharmonic oscillator model similar to that considered here. They showed that owing to periodicity the superposition coefficients for arbitrary N can be written down explicitly. We take advantage of this possibility in this paper.

First, it is easy to note that $|\psi(\tau + T)\rangle = |\psi(\tau)\rangle$ for $T = 2\pi$ because $n(n-1)$ is an even number. This means that the evolution is periodic in time (or length of the medium) with period $T = 2\pi$. Moreover, we have

$$\exp[i(\tau/2)(n+2N)(n+2N-1)] = \exp[i(\tau/2)n(n-1)]\exp[i\tau N(2N+2n-1)], \quad (10)$$

which means that for

$$\tau = (M/N)2\pi = (M/N)T \quad (11)$$

the exponential becomes periodic with period $2N$. We assume that M and N are mutually prime integers. When τ is taken as a fraction of the period, according to Eq. (11), then the state (9) becomes a superposition of coherent states[3]:

$$\left|\psi\left(\tau = \frac{M}{N}T\right)\right\rangle = \sum_{k=0}^{2N-1} c_k|\exp(i\varphi_k)\alpha_0\rangle, \quad (12)$$

where $|\alpha_0\rangle$ is the initial coherent state. The phases φ_k are given by

$$\varphi_k = (\pi/N)k, \qquad k = 0,1,\ldots,2N-1, \quad (13)$$

and the coefficients c_k are given by the set of $2N$ equations

$$\sum_{k=0}^{2N-1} c_k \exp(in\varphi_k) = \exp\left[i\pi\frac{M}{N}n(n-1)\right], \qquad n = 0,1,\ldots,2N-1. \quad (14)$$

Equation (14) can be rewritten as

$$\sum_{k=0}^{2N-1} c_k \exp\left\{i\frac{\pi}{N}[nk - Mn(n-1)]\right\} = 1, \quad (15)$$

which, after a summation over n and a minor rearrangement, gives

$$\sum_{k=0}^{2N-1} c_k \frac{1}{2N}\sum_{n=0}^{2N-1}\exp\left\{i\frac{\pi}{N}[nk - Mn(n-1)]\right\} = 1. \quad (16)$$

In view of the condition

$$\sum_{k=0}^{2N-1} c_k c_k^* = 1, \quad (17)$$

we immediately obtain

$$c_k = \frac{1}{2N}\sum_{n=0}^{2N-1} \exp\left\{-i\frac{\pi}{N}[nk - Mn(n-1)]\right\}. \quad (18)$$

Equation (18) gives the coefficients c_k of the superposition (12) for arbitrary M and N. Because of the symmetry of the system, only one half of the coefficients c_k are different from zero,[12] and the superposition (12) has only N components although the summation contains $2N$ terms. Anticipating this, we have extended the summations twice in order to preserve N for the number of components. Thus the denominator of the fraction M/N in Eq. (11) determines the number of components that appear in the superposition (12), which will contain the components with even (or odd) index only. Examples of such states are given in Ref. 12. Coefficients (18) can be rewritten in a different form,

$$c_k = \frac{1 + (-1)^{k-M(N-1)}}{2N} \times \sum_{n=0}^{N-1} \exp\left\{-i\frac{\pi}{N}[nk - Mn(n-1)]\right\}, \quad (19)$$

which explicitly shows that all c_k for which $k - M(N-1)$ is an odd number are equal to zero. That is, for $M(N-1)$ odd (even), only the coefficients with odd (even) k survive. The coefficients of the superposition have their modules equal to $1/\sqrt{N}$, and thus they can be written as

$$c_k = (1/\sqrt{N})\exp(i\gamma_k), \quad (20)$$

where the phases γ_k can be formally found from the relation

$$\gamma_k = -i\ln(\sqrt{N}c_k), \quad (21)$$

with c_k given by Eq. (19). The sums (18) defining the coefficients c_k are examples of trigonometric sums, which for some special cases can be summed exactly. It is not difficult, however, to calculate c_k numerically according to Eq. (19) and then to find γ_k from Eq. (21). This enables us to write down the superposition states explicitly for given M and N. Examples of such states will be illustrated graphically in Section 3 with plots of their quasi-probability distribution function $Q(\alpha, \alpha^*)$ as well as their phase distribution function $P(\theta)$.

3. QUASI-PROBABILITY DISTRIBUTION $Q(\alpha, \alpha^*)$ VERSUS PHASE DISTRIBUTION $P(\theta)$ AS A REPRESENTATION OF THE SUPERPOSITION STATES

The quasi-probability distribution $Q(\alpha, \alpha^*)$ is considered as a good representation of the quantum state of the field, and it has been used by several authors[8–12] to describe the anharmonic oscillator states. Miranowicz *et al.*[12] showed that this function reveals in a spectacular way the superpositions of coherent states that appear in the course of evolution of the anharmonic oscillator whenever the component states of the superposition are well separated. Recently Gantsog and Tanaś[13] showed that, as an alternative to the quasi-probability distribution $Q(\alpha, \alpha^*)$, the phase distribution $P(\theta)$ can be applied as a good indicator of the superpositions of coherent states. Although there is no direct simple relation between the two functions, they nevertheless have much in common in a description of the superpositions of coherent states. It is the aim of this paper to show explicitly the similarities.

The quasi-probability distribution $Q(\alpha, \alpha^*)$ is defined as[8]

$$Q(\alpha, \alpha^*, \tau) = \mathrm{Tr}[\hat{\rho}(\tau)|\alpha\rangle\langle\alpha|] = \langle\alpha|\hat{\rho}(\tau)|\alpha\rangle, \quad (22)$$

and it satisfies the relations

$$\int Q(\alpha, \alpha^*, \tau)\frac{d^2\alpha}{\pi} = 1, \quad (23)$$

$$0 \le Q(\alpha, \alpha^*, \tau) \le 1. \quad (24)$$

The properties of this function for the anharmonic oscillator states were discussed by Milburn[8] and Milburn and Holmes.[9]

For the initial coherent state $|\alpha_0\rangle$ the Q function has the form

$$Q(\alpha, \alpha^*, 0) = \exp(-|\alpha - \alpha_0|^2), \quad (25)$$

which represents a Gaussian bell centered on α_0.

For $\tau \neq 0$ the state of the field is given by Eq. (9), and the corresponding quasi-probability distribution is given by[8,10]

$$Q(\alpha, \alpha^*, \tau) = \langle\alpha|\psi(\tau)\rangle\langle\psi(\tau)|\alpha\rangle = \left|\sum_{n=0}^{\infty} \exp\left(-\frac{|\alpha|^2}{2}\right)\frac{|\alpha|^n}{\sqrt{n!}}b_n \times \exp\left\{i\left[n(\varphi_0 - \varphi) + \frac{\tau}{2}n(n-1)\right]\right\}\right|^2, \quad (26)$$

where the b_n are given by Eq. (8) and we have used the relation

$$\alpha = |\alpha|\exp(i\varphi). \quad (27)$$

It is evident from Eq. (26) that the quasi-probability distribution is periodic in τ; i.e.,

$$Q(\alpha, \alpha^*, \tau + T) = Q(\alpha, \alpha^*, \tau), \quad (28)$$

with period $T = 2\pi$.

For the values of τ given by Eq. (11) the state of the field becomes the superposition of coherent states given by Eq. (12), and in this case the quasi-probability distribution can be alternatively written as

$$Q\left(\alpha, \alpha^*, \tau = \frac{M}{N}T\right) = \left|\left\langle\alpha|\psi\left(\tau = \frac{M}{N}T\right)\right\rangle\right|^2 = \left|\sum_{k=0}^{2N-1} c_k\langle\alpha|\exp(i\varphi_k)\alpha_0\rangle\right|^2 = \left|\sum_{k=0}^{2N-1} c_k \exp\left\{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha_0|^2 + |\alpha||\alpha_0| \times \exp[i(\varphi_k + \varphi_0 - \varphi)]\right\}\right|^2, \quad (29)$$

where the coefficients c_k are given by Eq. (19). After minor rearrangement Eq. (29) takes the form

$$Q\left(\alpha, \alpha^*, \tau = \frac{M}{N}T\right) = \sum_{k,l=0}^{2N-1} c_k c_l^* \exp[-(|\alpha| - |\alpha_0|)^2] \times \exp\left\{-2|\alpha||\alpha_0| \times \left[1 - \cos\left(\frac{\varphi_k + \varphi_l}{2} + \varphi_0 - \varphi\right) \times \cos\left(\frac{\varphi_k - \varphi_l}{2}\right) - i \cos\left(\frac{\varphi_k + \varphi_l}{2} + \varphi_0 - \varphi\right)\sin\left(\frac{\varphi_k - \varphi_l}{2}\right)\right]\right\}. \tag{30}$$

It is clear from Eq. (30) that for $|\alpha_0| > 1$ the first exponential on the right-hand side is essentially different from zero only for $|\alpha| \simeq |\alpha_0|$, which means that the maxima of the Q function are located on the circle of radius $|\alpha_0|$. But even for $|\alpha| = |\alpha_0|$ the second exponential in Eq. (30) gives nonnegligible values for definite phase relations only. The maximum is obtained for $\varphi_k = \varphi_l$ and $\varphi = \varphi_0 + \varphi_k$. The condition $\varphi_k = \varphi_l$ refers to the contribution from the kth coherent component only. Equation (30) can be generally separated into two parts,

$$Q\left(\alpha, \alpha^*, \tau = \frac{M}{N}T\right) = Q_0 + Q_{\text{int}}, \tag{31}$$

where

$$Q_0 = \sum_{k=0}^{2N-1} |c_k|^2 \exp[-(|\alpha| - |\alpha_0|)^2] \times \exp\{-2|\alpha||\alpha_0|[1 - \cos(\varphi_k + \varphi_0 - \varphi)]\} \tag{32}$$

is the sum of the Gaussian quasi-probability distributions for individual coherent states of the superposition and

$$Q_{\text{int}} = \sum_{\substack{k,l=0 \\ k\neq l}}^{2N-1} c_k c_l^* \exp[-(|\alpha| - |\alpha_0|)^2]\exp\left\{-2|\alpha||\alpha_0| \times \left[1 - \cos\left(\frac{\varphi_k + \varphi_l}{2} + \varphi_0 - \varphi\right) \times \cos\left(\frac{\varphi_k - \varphi_l}{2}\right) - i \cos\left(\frac{\varphi_k + \varphi_l}{2} + \varphi_0 - \varphi\right) \times \sin\left(\frac{\varphi_k - \varphi_l}{2}\right)\right]\right\} \tag{33}$$

describes the interference terms. Whenever the interference terms are negligible, the quasi-probability distribution splits into separate peaks representing individual coherent states.[12] The number of well-distinguishable states in the superposition is proportional to $|\alpha_0|$. To estimate this number, we require negligible values of the interference terms for adjacent states. According to Eq. (13), $\varphi_{k+1} - \varphi_k = \pi/N$, and the interference terms will be negligible for

$$2|\alpha_0|^2\left[1 - \cos^2\left(\frac{\pi}{2N}\right)\right] > 1 \tag{34}$$

or

$$N < (\pi/\sqrt{2})|\alpha_0| \simeq 2.21|\alpha_0|. \tag{35}$$

If the number of components in the superposition satisfies inequality (35), the quasi-probability distribution for such states is described by Eq. (32) and has a quite regular structure, which is clearly seen from Fig. 1, where contours of the exact Q function given by Eq. (30) are compared with contours of Q_0 given by Eq. (32). The contours are obtained for the section at one half of the maximum height of the exact Q function. The solid curves are the contours of the exact Q function, and the dashed curves are those of the Q_0 function. For the maximum number of well-distinguishable states, which according to inequality (35) is in this case equal to 4, the four-peak structure of the Q function is still clearly visible, although there are already some differences between the contours of the exact Q function and those of their Gaussian part Q_0. This means that the interference terms (33) start to play a role.

Another representation of the field state that nicely indicates the superpositions of coherent states is the phase probability distribution $P(\theta)$, which is now available owing to the Hermitian phase formalism of Pegg and Barnett.[14–16] This formalism is based on the introduction of a finite $(s + 1)$-dimensional subspace Ψ spanned by the number states $|0\rangle, |1\rangle, \ldots, |s\rangle$. The Hermitian phase operator acts on this finite subspace, and after all necessary expectation values have been calculated in Ψ, the value of s is permitted to tend to infinity. A complete orthonormal basis of $(s + 1)$ states is defined on Ψ as

$$|\theta_m\rangle \equiv (s + 1)^{-1/2} \sum_{n=0}^{s} \exp(in\theta_m)|n\rangle, \tag{36}$$

where

$$\theta_m \equiv \theta_0 + 2\pi m/(s + 1), \qquad m = 0, 1, \ldots, s. \tag{37}$$

The value of θ_0 is arbitrary and defines a particular basis set of $s + 1$ mutually orthogonal phase states. The Hermitian phase operator is defined as

$$\hat{\phi}_\theta \equiv \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|. \tag{38}$$

The phase states (36) are, of course, eigenstates of the phase operator (38), with the eigenvalues θ_m restricted to lie within a phase window of θ_0 and $\theta_0 + 2\pi$.

We are interested in phase properties of the anharmonic oscillator states given by Eq. (9) or in the case when they become a superposition of coherent states given by Eq. (12). The probability amplitude for the phase taking the value θ_m is thus given by

$$\langle\theta_m|\psi(\tau)\rangle = (s + 1)^{-1/2} \times \sum_{n=0}^{s} b_n \exp\left\{i\left[n(\varphi_0 - \theta_m) + \frac{\tau}{2}n(n - 1)\right]\right\}, \tag{39}$$

and for the phase probability we obtain

$$|\langle\theta_m|\psi(\tau)\rangle|^2 = \frac{1}{s + 1} + \frac{2}{s + 1} \times \sum_{n>k} b_n b_k \cos\left\{(n - k)(\varphi_0 - \theta_m) + \frac{\tau}{2} \times [n(n - 1) - k(k - 1)]\right\}. \tag{40}$$

In the limit as s tends to infinity the continuous phase variable can be introduced,[14–16] and as a result the continuous phase probability distribution given by the following formula is obtained[13]:

$$P(\theta) = \frac{1}{2\pi}\left(1 + 2\sum_{n>k} b_n b_k \cos\left\{(n-k)(\varphi_0 - \theta) + \frac{\tau}{2}[n(n-1) - k(k-1)]\right\}\right), \tag{41}$$

with the normalization

$$\int_{\theta_0}^{\theta_0+2\pi} P(\theta)\mathrm{d}\theta = 1. \tag{42}$$

For $\tau = 0$ Eq. (41) describes the phase probability distribution for a coherent state $|\alpha_0\rangle$—a member of partial phase states. In this case the phase distribution $P(\theta)$ is peaked at $\theta = \varphi_0$, where φ_0 is the phase of α_0. Phase properties of the anharmonic oscillator states for $\tau \neq 0$ were discussed by Gantsog and Tanaś.[13] Here we are interested only in cases of special τ values given by Eq. (11), namely, those for which discrete superpositions of coherent states appear. In such cases the state of the field can be written in the form (12), and the phase probability distribution takes the form

$$P(\theta) = \frac{1}{2\pi}\sum_{k,l=0}^{2N-1} c_k c_l^* \sum_{n,n'} b_n b_{n'} \times \exp[in(\varphi_k + \varphi_0 - \theta) - in'(\varphi_l + \varphi_0 - \theta)]. \tag{43}$$

Similarly, as for the Q function, we can split $P(\theta)$ into two parts:

$$P(\theta) = P_0(\theta) + P_{\mathrm{int}}(\theta), \tag{44}$$

where

$$P_0(\theta) = \frac{1}{2\pi}\sum_{k=0}^{2N-1} |c_k|^2 \left\{1 + 2\sum_{n>n'} b_n b_{n'} \times \cos[(n-n')(\varphi_k + \varphi_0 - \theta)\right\} = \sum_{k=0}^{2N-1} |c_k|^2 P_k(\theta) \tag{45}$$

describes the sum of phase distributions of individual coherent states forming the superposition and

$$P_{\mathrm{int}}(\theta) = \frac{1}{2\pi}\sum_{k\neq l}^{2N-1} c_k c_l^* \sum_{n,n'} b_n b_{n'} \times \exp[in(\varphi_k + \varphi_0 - \theta) - in'(\phi_l + \varphi_0 - \theta)] \tag{46}$$

describes the interference terms contributing to the phase distribution.

Comparing Eqs. (45) and (46) with Eqs. (32) and (33), one easily finds that the maxima of the Q function appear for $\varphi = \varphi_k + \varphi_0$ and the maxima of $P(\theta)$ appear for $\theta = \varphi_k + \varphi_0$. This means that for well-separated states, when the interference terms are negligible, the two functions should have the same rotational symmetry. That this is really so is convincingly seen by a comparison of Figs. 1 and 2, where we plot, respectively, the contours of the Q function in the complex α plane and the phase probability distribution $P(\theta)$ versus θ in the polar coordinate system, each for several cases of the superpositions with well-separated component states. The solid curves correspond to the exact results, and the dashed curves correspond to the results obtained according to the simplified formulas (32) and (45). It is seen that the simplified formulas reproduce quite well the exact results. The results illustrated in Figs. 1 and 2 are obtained for $|\alpha_0| = 2$, and even for $N = 4$, i.e., for the upper limit of the

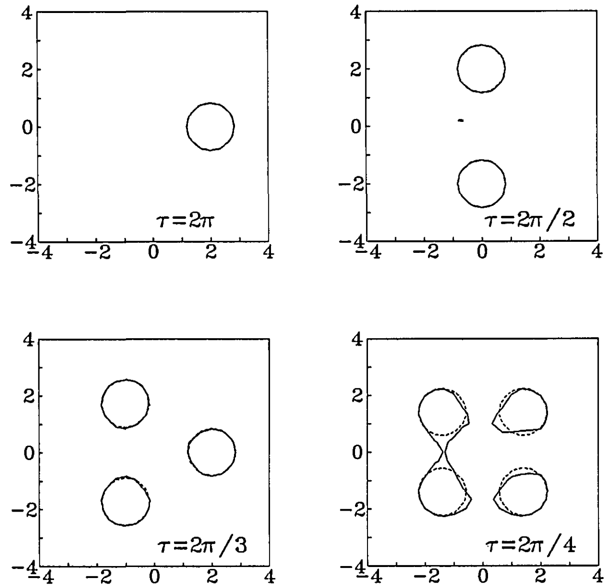

Fig. 1. Contours of the exact Q and Q_0 quasiprobability distributions in the complex α plane for the discrete superpositions of coherent states with N = 1–4 components. The solid curves correspond to the exact function Q, and the dashed curves correspond to Q_0. The phase φ_0 is taken to be zero everywhere.

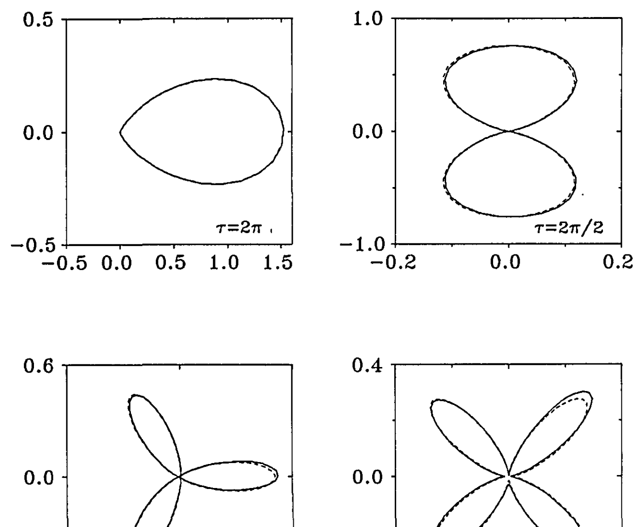

Fig. 2. Plots of the phase distribution $P(\theta)$ in the polar coordinates for the discrete superpositions of coherent states with N = 1–4 components. The solid curves correspond to exact results, and the dashed curves correspond to $P_0(\theta)$.

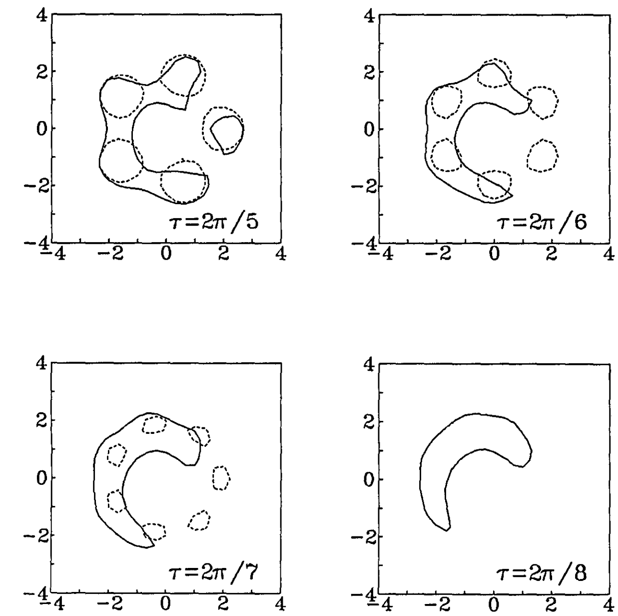

Fig. 3. Same as Fig. 1, but for N = 5–8.

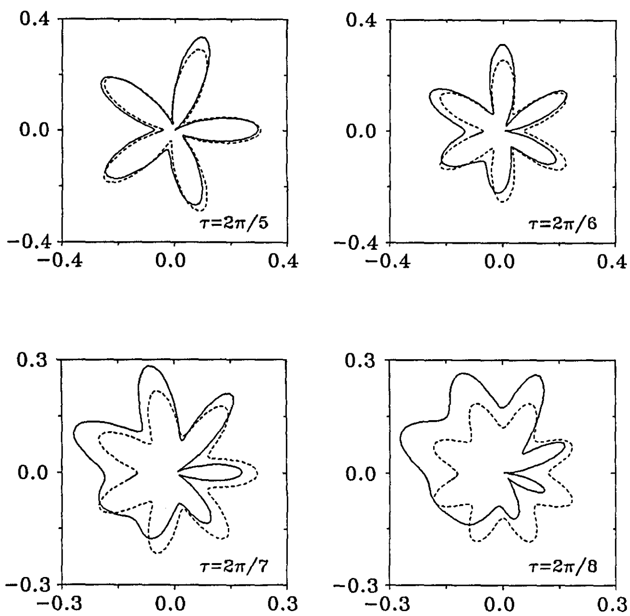

Fig. 4. Same as Fig. 2, but for N = 5–8.

estimate (35) the states are well separated. So the limit $N_{\max} \simeq 2|\alpha_0|$ can be considered as a good estimate of the maximal number of the well-separated states in the superposition. The estimate (35) agrees with the earlier estimates,[12] which were based on a different idea.

In Figs. 3 and 4 the increasing role of the interference terms in both the Q and P functions is illustrated. Solid curves again are used to plot the two functions according to the exact formulas [Eq. (26) with $\tau = (M/N)T$ for the Q function and Eq. (41) for the P function]. The phase φ_0 of the initial coherent state is taken everywhere as zero. It is evident from Figs. 3 and 4 that as the number of components in the superposition becomes larger than the maximal number of well-separated states ($N_{\max} = 4$ in this case), the interference terms play an increasing role and the symmetry of both the Q and P functions is destroyed.

Since the contours of the Q and Q_0 functions are obtained for the section at one half of the maximum height of the exact Q function for both Q and Q_0 contours, and since the peaks of the Q_0 function are proportional to $1/N$, the contours of the Q_0 function with regular N-fold symmetry become smaller as N increases, and they disappear for $N = 8$ because they are already below the section plane. For $N = 8$ the crescent shape of the contour[10] is recovered.

The same rotational symmetry of the Q and P functions for the well-separated states comes from the fact that the phases φ_k of the component states given by Eq. (13) take values that are a fraction k/N of π, where k and N are integers. Because every second coefficient c_k is equal to zero, they in fact divide 2π into N parts, forming the N-fold symmetry of both the Q and P functions. The similarity of the phase dependence of the Q function and the phase distribution $P(\theta)$ in the case of the anharmonic oscillator model can be shown in the following way. Let us make the transition to polar coordinates r and φ [$\alpha = r\exp(i\varphi)$] in the Q function defined by Eq. (26) and integrate it over r. This gives

$$\begin{aligned} Q(\varphi) &= \frac{1}{\pi}\int_0^\infty Q(r,\varphi) r\,\mathrm{d}r \\ &= \exp(-|\alpha_0|^2)\sum_{n,k=0}^{\infty}\frac{|\alpha_0|^{n+k}}{n!k!}\exp\left\{-i(n-k)\right. \\ &\quad\times\left.\left[\varphi-\varphi_0-\frac{\tau}{2}(n+k-1)\right]\right\} \\ &\quad\times\frac{1}{\pi}\int_0^\infty r^{n+k+1}\exp(-r^2)\mathrm{d}r \\ &= \frac{1}{2\pi}\left(1+2\sum_{n>k} b_n b_k \frac{\Gamma(n+k/2+1)}{\sqrt{n!k!}}\right. \\ &\quad\times\cos\left\{(n-k)(\varphi_0-\varphi)\right. \\ &\quad\left.\left.+\frac{\tau}{2}[n(n-1)-k(k-1)]\right\}\right). \end{aligned} \tag{47}$$

Expression (47) shows a striking similarity to the phase probability distribution (41). The only difference is the presence in Eq. (47) of the factor $\Gamma(n+k/2+1)/\sqrt{n!k!}$. This expression is valid for any τ values, not only for $\tau = (M/N)T$, i.e., for the superposition states. The function $Q(\varphi)$, which is properly normalized, could be considered as a phase distribution, but it differs from the true phase distribution (41). The differences may not be substantial in practice, so, at least for the anharmonic oscillator states, there is a deeper relation between the Q and P functions.

4. CONCLUSION

In this paper we have compared two representations of quantum states of the field. The quasi-probability distribution $Q(\alpha,\alpha^*)$ and the phase probability distribution $P(\theta)$ have been analyzed for the quantum states generated in the anharmonic oscillator model. For special choices of the evolution time the anharmonic oscillator states become superpositions of coherent states. It has been shown that for the superpositions with well-distinguishable states the Q and P functions have the same rotational symmetry. Both functions split into the sums of their counterparts for the individual components of the superposition, which may be of value for detecting the superposition states. The maximum number of well-distinguishable states in the superposition is estimated to be proportional to $|\alpha_0|$. The results were illustrated graphically, showing explicitly the rotational symmetry of the two functions for well-separated states and the growing influence of the interference terms when the number of components becomes larger than $N_{\max}$. The more fundamental relation between the phase dependence of the Q function and the phase probability distribution $P(\theta)$ was shown to exist for the anharmonic oscillator states.

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