Coherent states in a finite-dimensional Hilbert space

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(Received 22 April 1994)

We construct, according to Glauber's definition, the coherent states in a univocally specified finite-dimensional Hilbert space. Their analytical form is obtained as an analytical solution of the problem raised by Bužek *et al.* [Phys. Rev. A **45**, 8079 (1992)]. The phase properties of the coherent states are studied within the Pegg-Barnett formalism.

PACS number(s): 42.50.Dv

Recently, Bužek *et al.* [1] discussed the analog of Glauber's coherent states in a finite-dimensional Hilbert space. The problem turned out to be difficult. A general analytical form of the finite-dimensional coherent states (FDCS), i.e., states of a harmonic oscillator in the finite-dimensional Hilbert space $\mathcal{H}_s$ spanned by $(s+1)$ number states [2], was not found in Ref. [1]. Nevertheless, they proposed a numerical analysis of the FDCS differing essentially from the ordinary Glauber coherent states, i.e., those defined in infinite-dimensional Hilbert space. In this paper we present a method enabling us to obtain the FDCS in closed analytical form, as the solution of the problem proposed by Bužek *et al.* [1].

Glauber [3] defined a coherent state in infinite-dimensional Hilbert space $\mathcal{H}$ applying a displacement operator $\hat{D}(\alpha,\alpha^*)$ on the vacuum state $|0\rangle$:

$$|\alpha\rangle = \hat{D}(\alpha,\alpha^*)|0\rangle, \tag{1}$$

where

$$\hat{D}(\alpha,\alpha^*) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}). \tag{2}$$

The definition (1) is usually applied to construct coherent states in various finite-dimensional state spaces [1,4], e.g., the FDCS here discussed.

Formally, a coherent state $|\alpha\rangle_s$ spanned by $(s+1)$ number-state vectors, can be expressed as follows:

$$|\alpha\rangle_{(s)} = \sum_{n=0}^{s} C_n^{(s)}|n\rangle, \tag{3}$$

with the normalization condition

$$_{(s)}\langle\alpha|\alpha\rangle_{(s)} = \sum_{n=0}^{s}|C_n^{(s)}|^2 \ = \ 1 \tag{4}$$

strictly fulfilled for arbitrary s. Hence the main task resides in finding the coefficients $C_n^{(s)}$. The Baker-Hausdorf formula cannot be used to solve this problem because the commutator of the annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators is not a c number. A numerical procedure, leading to the coefficients $C_n^{(s)}$, was proposed by Bužek *et al.* [1].

In order to solve this problem analytically, it is of advantage to express the coherent state $|\alpha\rangle$ in number-state representation in a different manner:

$$\begin{aligned}|\alpha\rangle &= \sum_{n=0}^{\infty}\frac{(\alpha a^\dagger - \alpha^* a)^n}{n!}|0\rangle \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[[n/2]]}\frac{\sqrt{(n-2k)!}}{n!}d_{n,n-2k}(-\alpha^*)^k\alpha^{n-k}|n-2k\rangle,\end{aligned} \tag{5}$$

where

$$d_{n,k} \equiv d_{n,k}^{(\infty)} \ = \ \binom{n}{k}(n-k-1)!!, \tag{6}$$

and $[\![x]\!]$ denotes integer $\le x$. In the s-dimensional Hilbert space, the condition

$$\hat{a}^{\dagger k}|n\rangle = 0 \qquad \text{for } n+k > s, \tag{7}$$

should be fulfilled. Thus the equation (5) for the FDCS can be rewritten as

$$|\alpha\rangle_{(s)} = \sum_{k=0}^{s}\sum_{n=k}^{\infty}\frac{\sqrt{k!}}{n!}d_{nk}^{(s)}(-\alpha^*)^{(n-k)/2}\alpha^{(n+k)/2}|k\rangle. \tag{8}$$

The problem reduces to the derivation of the coefficients $d_{n,k}^{(s)}$ in the $(s+1)$-dimensional space satisfying the limit condition

$$\lim_{s\to\infty} d_{nk}^{(s)} = d_{nk}^{(\infty)} \ \equiv \ d_{nk} \ = \ \binom{n}{k}(n-k-1)!! \tag{9}$$

We obtained the following simple recurrence formula for d_{nk} (more details in [5]):

$$d_{nk}^{(s)} = \theta_k d_{n-1,k-1}^{(s)} + (k+1)\theta_{k+1}d_{n-1,k+1}^{(s)} \tag{10}$$

with the boundary conditions for arbitrary s:

$$\begin{aligned}d_{00}^{(s)} &= 1, \\ d_{n,n+k}^{(s)} &= 0 \ \text{ for } k > 0,\end{aligned} \tag{11}$$

where the Heaviside function θ_n is defined as

$$\theta_n \equiv \theta(s-n) = \begin{cases}1 & \text{for } s \ge n \\ 0 & \text{for } s < n.\end{cases} \tag{12}$$

We arrive at the following solution of the recurrence formula (12):

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$$d_{nk}^{(s)} = \frac{s!}{k!(s+1)} \sum_{l=0}^{s} \frac{\mathrm{He}_k(x_l)}{[\mathrm{He}_s(x_l)]^2} x_l^n, \tag{13}$$

where $x_l \equiv x_l^{(s+1)}$ are the roots of the modified Hermite polynomial of order $(s+1)$,

$$\mathrm{He}_{s+1}(x_l) = 0. \tag{14}$$

A solution similar to ours (13), with the roots x_k (14), was found by Figurny *et al.* [6] in their analysis of the eigenvalues of the truncated (i.e., finite-dimensional) quadrature operators.

Our procedure provides the coefficients $C_n^{(s)}$ of (3) in the closed analytical form

$$C_n^{(s)} = \frac{s!}{s+1}(n!)^{-1/2} \sum_{k=0}^{s} \exp\{\mathrm{i}[n(\phi_0 - \pi/2) + x_k\,|\alpha|]\} \times \mathrm{He}_n(x_k)\,[\mathrm{He}_s(x_k)]^{-2}, \tag{15}$$

after performing summation in (5) with the coefficients $d_{nk}^{(s)}$ given by (13).

The expression (15) is the solution of the problem formulated by Bužek *et al.* [1]. Some properties of these states and more details concerning our procedure will be published shortly [5].

The FDCS (3), with the coefficients $C_n^{(s)}$ (15), take the following simple forms in the special cases for s =1,2,3:

$$|\alpha\rangle_{(1)} = \cos|\alpha||0\rangle + e^{i\phi_0}\sin|\alpha||1\rangle, \tag{16}$$

$$|\alpha\rangle_{(2)} = \frac{1}{3}\left[\cos\left(\sqrt{3}|\alpha|\right) + 2\right]|0\rangle + \frac{1}{\sqrt{3}}e^{i\phi_0}\sin\left(\sqrt{3}|\alpha|\right)|1\rangle + \frac{\sqrt{2}}{3}e^{2i\phi_0}\left[1 - \cos\left(\sqrt{3}|\alpha|\right)\right]|2\rangle, \tag{17}$$

$$\begin{aligned}|\alpha\rangle_{(3)} = {} & \frac{1}{2}\left[\frac{1}{x_1^2}\cos(x_1|\alpha|) + \frac{1}{x_2^2}\cos(x_2|\alpha|)\right]|0\rangle + \frac{1}{2}e^{i\phi_0}\left[\frac{1}{x_1}\sin(x_1|\alpha|) + \frac{1}{x_2}\sin(x_2|\alpha|)\right]|1\rangle \\ & - \frac{1}{2\sqrt{3}}e^{2i\phi_0}\left[\cos(x_1|\alpha|) - \cos(x_2|\alpha|)\right]|2\rangle - \frac{1}{2}e^{3i\phi_0}\left[\frac{1}{x_1}\sin(x_1|\alpha|) - \frac{1}{x_2}\sin(x_2|\alpha|)\right]|3\rangle,\end{aligned} \tag{18}$$

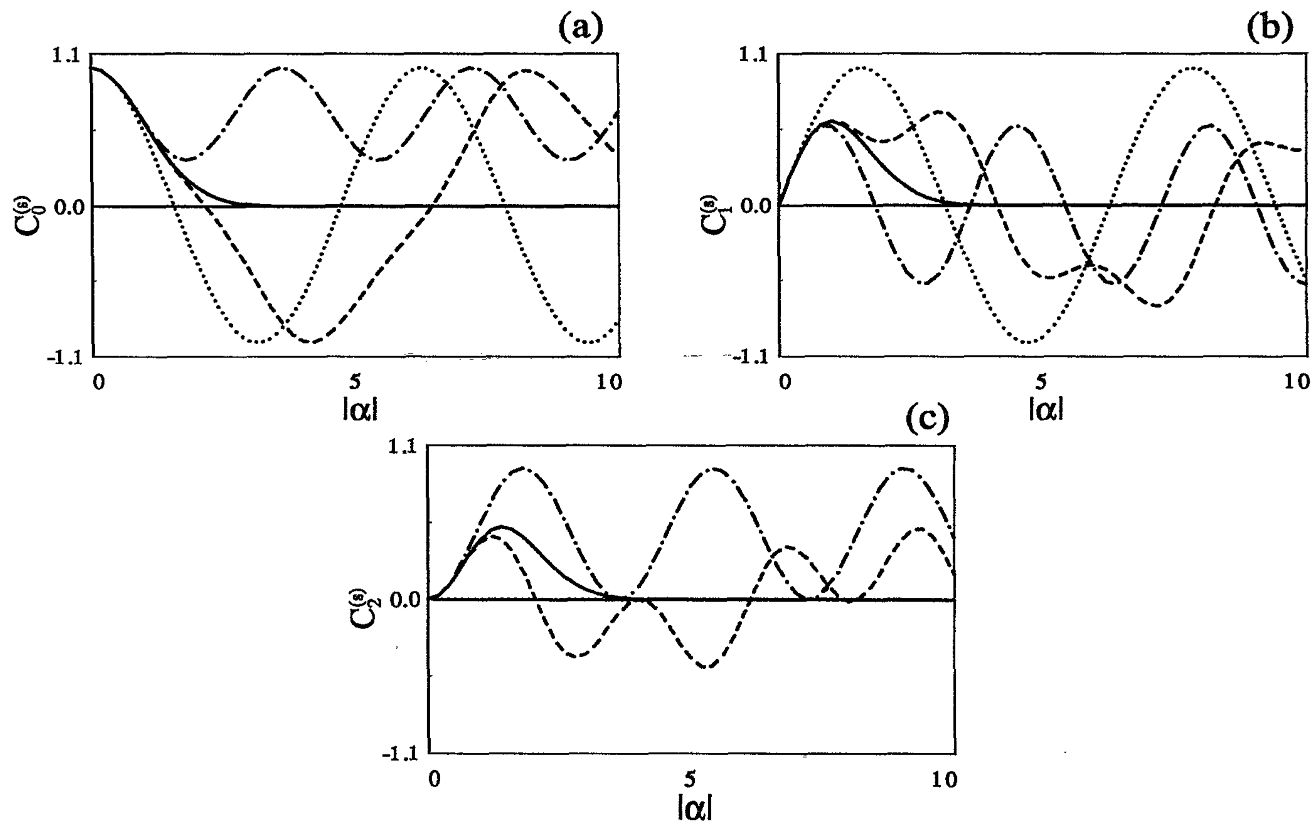

FIG. 1. The coefficients $C_n^{(s)} = |C_n^{(s)}|$ (15) plotted versus the amplitude $|\alpha|$ for (a) n=0, (b) n=1, (c) n=3 in the Hilbert spaces of different dimensionality: s = 1 (dotted lines), s = 2 (dotted-dashed lines), s = 3 (dashed lines), and $s = \infty$ (solid lines).

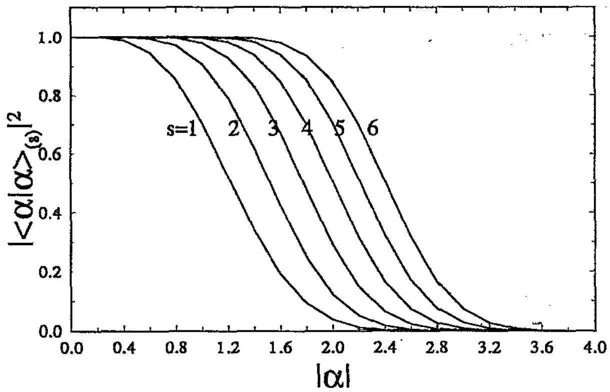

FIG. 2. The dependence of the scalar product of FDCS $|\alpha\rangle_s$ and infinite-dimensional coherent states $|\alpha\rangle$ on the amplitude $|\alpha|$ for various s.

where

$$\alpha = |\alpha| e^{i\phi_0}, \tag{19}$$

$$x_{1,2} \equiv x_{1,2}^{(4)} = \sqrt{3 \pm \sqrt{6}}. \tag{20}$$

The coherent state $|\alpha\rangle_{(1)}$ (16) in the two-dimensional space $\mathcal{H}_1$ spanned by $\{|0\rangle, |1\rangle\}$ was obtained and analyzed by Bužek *et al.* [1]. The simplicity of (16) comes from the fact that the only nonvanishing coefficients $d_{nk}^{(1)}$ are equal to unity.

In Fig. 1 the coefficients $C_n^{(s)}$ determining the photon-number distribution are plotted as functions of the amplitude $|\alpha| = \alpha$. It is clearly seen that the differences between $C_n^{(s)}$ and $C_n^{(\infty)}$, for $\alpha \lesssim \sqrt{n}$, vanish with increasing number of dimensions s. For $s = 1, 2$ the coefficients $C_n^{(s)}$ are given by a single trigonometric function multiplied and shifted by appropriate factors. The coefficient $C_n^{(3)}$, as depicted in Figs. 1(a)–1(c), is the superposition of two trigonometric functions. In general, the coefficients for the s-dimensional space are superpositions of $[\![\frac{s+1}{2}]\!]$ cosine functions.

In Fig. 2 we plot the squared absolute value of the scalar product $|\langle\alpha|\alpha\rangle_{(s)}|^2$ versus the amplitude α for particular values of s. Obviously, there is hardly any differference between our FDCS (3) and the Glauber coherent states (1) in the infinite-dimensional Hilbert space $\mathcal{H}$ for $\alpha/s \ll 1$. However, for $\alpha/s > 1$ the differences are significant.

Using the explicit form (15) one can easily analyze the phase properties of the FDCS within the Pegg-Barnett formalism [7,8]. Some of these properties have been studied in [1]. Here, we calculate the Pegg-Barnett distribution for FDCS defined as

$$P(\theta_m) \equiv \left|{}_{(s)}\langle\theta_m|\alpha\rangle_{(s)}\right|^2. \tag{21}$$

The FDCS are examples of partial phase states [7]. In this case it is convenient to choose the initial value θ_0 of the phase window as

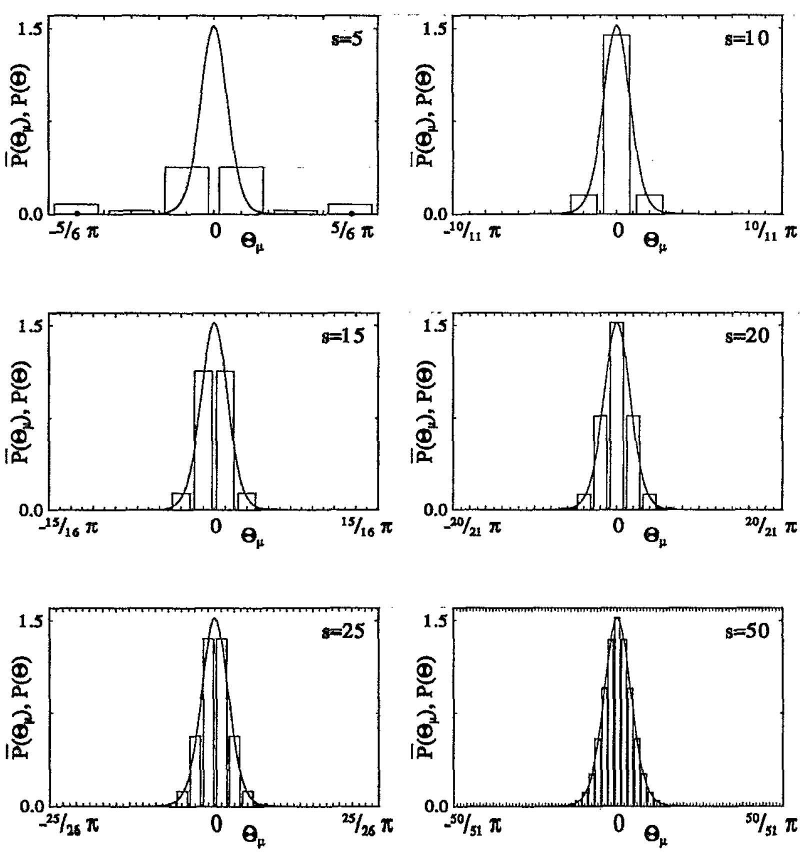

FIG. 3. The Pegg-Barnett distributions with discrete $\bar{P}(\theta_\mu)$ and continuous $P(\theta)$ phase dependence for $s = 5, ..., 50$.

$$\theta_0 = \phi_0 - \frac{s}{s+1}\pi \quad (22)$$

and renumber the dummy indices $\mu = m - s/2$, which leads to

$$\theta_\mu = \frac{2\pi}{s+1}\mu, \quad \mu = -\frac{s}{2}, \ldots, \frac{s}{2}. \quad (23)$$

Thus the Pegg-Barnett phase distribution (21) can be expressed in a form symmetrical with respect to μ:

$$P(\theta_\mu) = |_{(s)}\langle\theta_\mu|\alpha\rangle_{(s)}|^2 = \frac{1}{s+1}\left\{1 + 2\,\mathrm{Re}\sum_{n=1}^{s}\sum_{k=0}^{n-1}|C_n^{(s)}||C_k^{(s)}| \times \exp\left[i(n-k)\theta_\mu\right]\right\}. \quad (24)$$

In the limit of $s \mapsto \infty$ the continuous phase variable θ is introduced instead of θ_μ and $2\pi/(s+1)$ is replaced by $d\theta$. Hence one obtains the continuous Pegg-Barnett phase distribution in the following form:

$$\lim_{s\to\infty} \bar{P}(\theta_\mu) \equiv \lim_{s\to\infty} \frac{s+1}{2\pi} P(\theta_\mu) = P(\theta). \quad (25)$$

In Fig. 3 the discrete function $\bar{P}(\theta_\mu)$, i.e., the renormalized phase distribution $P(\theta_\mu)$ with the scaling factor $\frac{s+1}{2\pi}$ is depicted for the FDCS $|\alpha\rangle_{(s)}$ with $\alpha = 2$ in the finite-dimensional (s=5,...,50) Hilbert spaces. The normalization condition

$$\sum_{\mu=-s/2}^{s/2} P(\theta_\mu) = 1 \quad (26)$$

implies that the maximum of distributions $P(\theta_\mu)$ considerably decreases with increasing dimension s. This property makes a direct comparison of the discrete and continuous phase distributions difficult. To avoid this difficulty we have used the scaling factor $\frac{s+1}{2\pi}$ justified by the relation (25). It is clearly seen that the differences between the phase distributions $P(\theta_\mu)$ for FDCS and $P(\theta)$ for the coherent states in standard Hilbert space $\mathcal{H}$ vanish with increasing number s of dimensions of $\mathcal{H}_s$.

In conclusion, we have derived the analytical form of the coherent states in the finite-dimensional Hilbert space according to the general Glauber definition. We have tested the essential differences between the ordinary coherent states and ours as revealed by the photon-number and phase properties.

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