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PHASE COHERENT STATES[1]

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We analyze two different definitions of phase coherent states in a finite-dimensional Hilbert space. Their explicit phase-state expansions and their Wigner representation are given.

1. Introduction

Recently, Bužek, Wilson-Gordon, Knight and Lai [1] proposed a definition of annihilation and creation operators of the phase quanta in a finite $(s+1)$-dimensional Hilbert space. These operators are in a close analogy to well-known number creation and annihilation operators. Their idea proved fruitful and several recent articles deal with the properties of various states generated by these operators, including phase coherent states [2,3,4] and displaced phase states [2]. Here, we study two kinds of phase coherent states associated with the Pegg-Barnett Hermitian optical phase formalism [5]. First states can be generated by the action of the generalized phase displacement operator. This definition of phase coherent states (PCS) is close to Glauber's idea and was applied by Gangopadhyay [2]. Second definition of phase coherent states is based on another formally designed phase "displacement" operator as proposed by Kuang and Chen [3,4]. We shall refer to these states as truncated phase coherent states (TPCS). We construct PCS and TPCS explicitly and derive their Wigner representation in a finite-dimensional Hilbert space. In particular, the states are compared by calculating their scalar product. Here, we present only a glimpse of our analysis. More details, illustrated with figures, shall be given elsewhere [6].

2. Phase creation and annihilation operators

Phase creation, $\hat{\phi}_\theta$, and annihilation, $\hat{\phi}_\theta^\dagger$, operators were introduced by Bužek et al. [1] with the help of the relation $\hat{\Phi}_\theta = \hat{\phi}_\theta^\dagger \hat{\phi}_\theta$ for the Pegg-Barnett Hermitian optical phase operator $\hat{\Phi}_\theta$ [5]. They are defined in a finite-dimensional Hilbert space $\mathcal{H}^{(s)}$, which is

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spanned by a complete orthonormal set of number states $|0\rangle$, $|1\rangle, \ldots, |s\rangle$ or, equivalently, by a set of phase states

$$|\theta_m\rangle = (s+1)^{-1/2}\sum_{n=0}^{s}\exp(in\theta_m)|n\rangle, \qquad m = 0,1,...,s, \tag{1}$$

where $\theta_m = \theta_0 + \frac{2\pi}{s+1}m$. The operators $\hat{\phi}_\theta^{\pm}$ are expressed in the polar form via the photon-number operator $\hat{N}$ and phase $\hat{\Phi}_\theta$, analogously to polar form of the operators $\hat{a}^{\pm}$ but with interchanged $\hat{N}$ and $\hat{\Phi}_\theta$. The phase annihilation operator $\hat{\phi}_\theta$, in the phase-state basis, has the following form

$$\hat{\phi}_\theta = \sum_{m=1}^{s}\sqrt{\theta_m}|\theta_{m-1}\rangle\langle\theta_m| \tag{2}$$

and the phase creation operator $\hat{\phi}_\theta^\dagger$ is simply a Hermitian conjugate of (2). Clearly, only for $\theta_0 = 0$, the phase annihilation operator $\hat{\phi}_\theta$ (2) in the phase-state basis has the same form as the (photon-number) annihilation operator $\hat{a}$ in the number-state basis. Besides, the operators $\hat{\phi}_\theta^{\pm}$ act on the phase states in a similar way (particularly for $\theta_0 = 0$) as the ordinary operators $\hat{a}^{\pm}$ act on Fock states [1].

3. Phase coherent states

Phase coherent states have been studied both in the Glauber [2] and Kuang-Chen [3,4] sense. The main idea is to choose a preferred phase state $|\theta_0\rangle$ and to refer to it as *phase vacuum*; and then to construct phase creation $\hat{\phi}_\theta^\dagger$ and phase annihilation $\hat{\phi}_\theta$ operators in analogy to the usual creation and annihilation operators. The phase coherent state is then constructed by replacing vacuum $|0\rangle$ by $|\theta_0\rangle$ and the operators $\hat{a}$, $\hat{a}^\dagger$ by $\hat{\phi}_\theta$, $\hat{\phi}_\theta^\dagger$ in the definition of coherent states. So, in the Glauber sense, the phase coherent states $|\beta,\theta_0\rangle_{(s)}$ for $\beta = |\beta|\exp(i\varphi)$ can be defined as $|\beta,\theta_0\rangle_{(s)} = \hat{D}^{(s)}(\beta,\theta_0)\,|\theta_0\rangle$, i.e., by the action of the finite-dimensional phase displacement operator $\hat{D}^{(s)}(\beta,\theta_0) = \exp(\beta\hat{\phi}_\theta^\dagger - \beta^*\hat{\phi}_\theta)$, which is given in terms of the phase creation and annihilation operators. This definition was proposed by Gangopadhyay [2]. Applying the method developed in [7], we have found the following phase-state representation of the phase coherent states, for various values of θ_0

$$|\beta,\theta_0\rangle_{(s)} = \sum_{m=0}^{s} e^{i(\mu-m_0)\varphi}\, b_m^{(s)}|\theta_m\rangle \tag{3}$$

with the decomposition coefficient

$$b_m^{(s)} \equiv b_m^{(s)}(\theta_0) = \frac{s!}{s+1}(-1)^{m+m_0-\mu}(\mu!m_0!)^{-1/2}\,i^{m_0}(-i)^{\mu} \times \sum_{k=0}^{s}\exp\left(ix_k\gamma_s|\beta|\right)\mathrm{He}_\mu(x_k)\mathrm{He}_{m_0}(x_k)\mathrm{He}_s^{-2}(x_k). \tag{4}$$

Here, $x_l \equiv x_l^{(s+1)}$ are the roots of the modified Hermite polynomial of order $(s+1)$, $\mathrm{He}_{s+1}(x_l) = 0$, and $\mathrm{He}_n(x) \equiv 2^{-n/2}\mathrm{H}_n(x/\sqrt{2})$. For brevity, we have denoted $\mu = m + m_0 \; mod(s+1)$ and $\gamma_s = \left(\frac{2\pi}{s+1}\right)^{1/2}$. The values θ_m are $mod(2\pi)$. We also assume that the permitted values of θ_0 are not completely arbitrary and are equal to $2\pi/(s+1)m_0 \; mod(2\pi)$ (where m=0,1,...). This is the main result of our paper. In a special case, for $\theta_0 = 0$ and $s = 1$, the PCS (3) reduces to the state $|\beta, \theta_0 = 0\rangle_{(1)}$ studied by Gangopadhyay [2]. Here, for simplicity, we consider only the case of $\theta_0 = 0$.

4. Truncated phase coherent states

Kuang and Chen [3,4] defined the phase coherent states $|\overline{\beta}, \theta_0\rangle_{(s)}$ in $\mathcal{H}^{(s)}$ by the action of the finite-dimensional operator $\exp(\overline{\beta}\hat{\phi}_\theta^\dagger)$ on the phase state $|\theta_0\rangle$. The reference phase θ_0 is chosen as zero [3,4]. Therefore, on comparing the explicit expressions for $\hat{a}$ and $\hat{\phi}_\theta$ (2), it is clear that the states $|\overline{\beta}, \theta_0\rangle_{(s)}$ are in close analogy to the truncated coherent states [10]. For this reason we shall refer to the states $|\overline{\beta}, \theta_0\rangle_{(s)}$ as *truncated phase coherent states* in $\mathcal{H}^{(s)}$. For completeness, we present their phase-space expansion explicitly for $\overline{\beta} = |\overline{\beta}| \exp(\mathrm{i}\varphi)$:

$$|\overline{\beta}, \theta_0\rangle_{(s)} = \mathcal{N}^{(s)} \exp(\overline{\beta}\hat{\phi}_\theta^\dagger)|\theta_0\rangle = \sum_{m=0}^{s} \mathrm{e}^{\mathrm{i}m\varphi}\, b_m^{(s)} |\theta_m\rangle, \tag{5}$$

$$b_m = \mathcal{N}^{(s)} (\gamma_s |\overline{\beta}|)^m (m!)^{-1/2}, \qquad \mathcal{N}^{(s)} = \left(\sum_{n=0}^{s} \frac{(\gamma_s|\overline{\beta}|)^{2n}}{n!}\right)^{-1/2}, \tag{6}$$

where $\mathcal{N}^{(s)}$ is the normalization. In particular, squeezing properties of the states (5) were analyzed by Kuang and Chen [3,4]. They have paid special attention to the two-dimensional case.

5. Discussion

Although many properties of the phase coherent states are known by now, for their better understanding it is very useful to analyze graphs of their quasidistributions. Here, we shall restrict our attention to some analytical expressions. A full analysis will be presented elsewhere. The discrete Wigner function, as defined by Wootters [8] (see also [9]), takes the following form for $s > 1$

$$W(n, \theta_m) = \frac{1}{s+1} \sum_{p=0}^{s} b_{m+p}\, b_{m-p} \exp\left[-2\mathrm{i}p\left(\frac{2\pi}{s+1}n + \varphi\right)\right] \tag{7}$$

for the PCS with b_n given by (4) and for the TPCS with superposition coefficients (6). In eq. (7), the subscripts $m \pm p$ are $mod(s+1)$. We have obtained the particularly simple Wigner function for $s = 1$ [8].

Phase coherent states $|\beta, \theta_0\rangle_{(s)}$ and truncated phase coherent states $|\overline{\beta}, \theta_0\rangle_{(s)}$ are associated with the Pegg-Barnett formalism of the Hermitian phase operator $\hat{\Phi}_\theta$. Since the operators $\hat{\Phi}_\theta$ and $\hat{\phi}_\theta^{\pm}$ do not exist in the usual (i.e., infinite-dimensional) Hilbert space $\mathcal{H}^{(\infty)}$, the PCS and TPCS are properly defined only in $\mathcal{H}^{(s)}$ of finite dimension. The phase coherent states $|\beta, \theta_0\rangle_{(s)}$ and truncated phase coherent states $|\overline{\beta}, \theta_0\rangle_{(s)}$, similarly to the Glauber coherent states $|\alpha\rangle_{(s)}$ and truncated coherent states $|\overline{\alpha}\rangle_{(s)}$ [10], approach each other for $|\beta|^2 = |\overline{\beta}|^2 \ll s/\pi$. It can be explicitly shown by calculating the scalar product between PCS and TPCS. We find ($\beta = \overline{\beta}$):

$$ {}_{(s)}\langle \beta, \theta_0 | \overline{\beta}, \theta_0 \rangle_{(s)} = 1 - \frac{(\sqrt{\pi}|\beta|)^{2(s+2)}}{2s!(s+2)^2} + \mathcal{O}(|\beta|^{2(s+3)}). \tag{8} $$

For values $|\beta|^2 = |\overline{\beta}|^2 \approx s/\pi$ or greater than s/π, the differences between $|\beta, \theta_0\rangle_{(s)}$ and $|\overline{\beta}, \theta_0\rangle_{(s)}$ become significant. Besides, we have shown in [6] (see also [10]) that PCS are periodic or quasi-periodic in β, whereas TPCS are aperiodic in $\overline{\beta}$ for any dimension.

The finite-dimensional phase coherent states, discussed here, are not only mathematical structures. A framework for their physical interpretation is provided by cavity quantum electrodynamics and atomic physics. Besides, they can be generated, e.g., in a single-mode resonator. Several methods have been proposed for preparation of an arbitrary field state (e.g., [11] and references therein), which can readily be applied for generation of these finite-dimensional states. Also, a scheme, developed by Leoński and Tanaś [12], seems to be very promising.

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