Laser Physics, Vol. 7, No. 1, 1997, pp. 126–130.

Quasi-Periodic and Periodic Evolution of Cavity Field in a Nonlinear Medium

W. Leoński, A. Miranowicz, and R. Tanaś

Institute of Physics, A. Mickiewicz University, Umultowska 85, Poznań, 61-614 Poland

e-mail: wleonski@phys.amu.edu.pl; miran@phys.amu.edu.pl; tanas@phys.amu.edu.pl

Received July 23, 1996

Abstract—"Time evolution" of coherent states defined in a finite-dimensional Hilbert space is considered. Such states can be generated using the model comprising a cavity filled with the nonlinear medium with the $(2q-1)$th nonlinearity $\chi^{(2q-1)}$. Two definitions of the finite-dimensional coherent states are considered: (i) Glauber-like coherent states and (ii) truncated coherent states. We concentrate our attention on the periodic and quasi-periodic features in the evolution of these states.

1. INTRODUCTION

The states that are most commonly used in quantum optics are Glauber coherent states [1]. These states are defined in an infinite-dimensional Hilbert space. Quite recently, Pegg and Barnett [2–4] have introduced a Hermitian phase operator defined in a finite-dimensional Hilbert space (FDHS), and finite-dimensional Hilbert spaces became themselves a subject of studies. Because of wide interest in FDHS, a number of authors tried to adapt the definition of Glauber coherent states to the finite-dimensional spaces. There are different possibilities to do this. In this paper we will concentrate on two of them. One definition is based on the treatment of the coherent states as a result of the action of the displacement operator on the vacuum state, where the displacement operator has the same form as in the Glauber definition [1], but it is spanned in the FDHS. This definition was applied and discussed by a number of authors, e.g., Bužek *et al.* [5] and Miranowicz *et al.* [6]. An alternative attempt to define coherent states in FDHS is based on the truncation of the number state decomposition of the ordinary coherent state (defined in the infinite-dimensional space) to a finite number of Fock states with properly normalized amplitudes. This method was proposed by Kuang *et al.* [7, 8]. In this paper, similarly to [9], the FDHS coherent states defined according to the first definition will be referred to as finite-dimensional Glauber coherent states (FDGCS), whereas the states defined according to the second one, as in [7, 8], will be called truncated coherent states (TCS). We will show that the two types of coherent states evolve differently. We will focus our attention on their quasi-periodic and periodic properties.

2. DEFINITIONS

In this paper we will consider the "time evolution" of coherent states defined in a finite-dimensional Hilbert space. Such states can be generated in a model comprising a cavity filled with the nonlinear medium exhibiting the $(2q-1)$th nonlinearity $\chi^{(2q-1)}$ [10]. This model is governed by the following Hamiltonian (in the interaction picture):

$$\hat{H} = \frac{\chi^{2q-1}}{q}(\hat{a}^{\dagger})^{q}\hat{a}^{q} + \epsilon(\hat{a}^{\dagger} + \hat{a}), \tag{1}$$

where ϵ denotes the strength of the coupling between the external driving field and the cavity field, and the driving field is assumed to be tuned exactly to the cavity frequency. Assuming that the external field excitation is weak, $\epsilon \ll \chi^{2q-1}$, and that the field inside the cavity is initially in the vacuum state $|0\rangle$, this system can generate a very specific state of the field. Namely, the time evolution of the system is restricted to a finite set of the first q number states. Applying the perturbation theory, we can derive the appropriate formulas for the amplitudes corresponding to the subsequent number states. For instance, for the case of $q = 3$ discussed in [10], we can write the solutions for the amplitudes in the form

$$\begin{aligned}
c_0(t) &= \frac{1}{3}(2 + \cos(\sqrt{3}\epsilon t)) + \mathcal{O}(\epsilon),\\
c_1(t) &= \frac{-i}{\sqrt{3}}\sin(\sqrt{3}\epsilon t) + \mathcal{O}(\epsilon),\\
c_2(t) &= \frac{\sqrt{2}}{3}(\cos(\sqrt{3}\epsilon t) - 1) + \mathcal{O}(\epsilon),\\
c_3(t) &= \frac{-2i\epsilon}{3\chi\sqrt{6}}(\cos(\sqrt{3}\epsilon t) - 1) + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{2}$$

It is seen that for weak external excitation the influence of the three-photon state $|3\rangle$ on the dynamics of the system is negligible. The amplitudes c_0, c_1, c_2 appearing in (2) correspond to the probability amplitudes derived in [6] for the coherent states defined in the three-dimensional space. Thus, with the high accuracy, we can treat the resulting state as an example of the coherent state in FDHS. The argument of the cosine and sine functions appearing in (2) is related to the parameter $|\alpha|$ appearing in the formulas for the finite-dimensional coherent state expansion derived in [6]. In fact, when we compare our formulas, given by (2), with those of [6] we find a simple relation between the parameter $|\alpha|$ appearing in [6] and the quantities appearing in (2), i.e., $|\alpha| = \epsilon t$. This relation can be helpful in the physical interpretation of $|\alpha|$ introduced in the definitions for finite-dimensional Hilbert spaces. We will refer to this relation in our discussion of "time evolution" of coherent states in FDHS.

From now on, we abandon the discussion concerning methods of generation of the FDHS coherent states, and we will concentrate on their periodical properties. In particular, we will compare the properties of the coherent states in FDHS defined in two different ways. As it has been mentioned earlier, Kuang *et al.* [7, 8] defined the coherent states in FDHS by the truncation of the Fock decomposition of the Glauber infinite-dimensional coherent state $|\alpha\rangle_{(\infty)}$. This is equivalent to the action of a nonunitary operator $\exp(\bar{\alpha}\hat{a}^{\dagger})$ on the vacuum state. Of course, proper normalization of the states is required. The states obtained in this way are referred to as truncated coherent states (TCS). We can rewrite the expansion of this state in the number states basis as in [9]:

$$|\bar{\alpha}\rangle_{(s)} = \mathcal{N}^{(s)}\exp(\bar{\alpha}\hat{a}^{\dagger})|0\rangle = \sum_{n=0}^{s}\exp(in\phi)b_n^{(s)}|n\rangle, \quad (3)$$

where

$$b_n^{(s)} = \mathcal{N}^{(s)}|\bar{\alpha}|^n(n!)^{-1/2}. \quad (4)$$

The normalization constant $\mathcal{N}$ appearing in (4) can be written in the following form:

$$\mathcal{N}^{(s)} = \left(\sum_{n=0}^{s}\frac{|\bar{\alpha}|^{2n}}{n!}\right)^{-1/2} = \{(-1)^s L_s^{-s-1}(|\bar{\alpha}|^2)\}^{-1/2}, \quad (5)$$

where the quantity L_s^{-s-1} is a generalized Laguerre polynomial. Obviously, the dimension of the Hilbert space is equal to $(s + 1)$.

The other type of FDHS coherent states are those referred to as finite-dimensional Glauber coherent states. They are defined, in the same way as the appropriate states in infinite-dimensional Hilbert space, by the action of the displacement operator on the vacuum, except for the fact that the Glauber unitary displacement operator $\exp(\alpha\hat{a}^{\dagger} - \alpha^*\hat{a})$ is defined in the FDHS. Thus, for the $(s + 1)$-dimensional Hilbert space, the expansion of the FDGCS can be written as follows [6]:

$$|\alpha\rangle_s = \exp(\alpha\hat{a}^{\dagger} - \alpha^*\hat{a})|0\rangle = \sum_{n=0}^{s}\exp(in\phi)c_n^{(s)}|n\rangle, \quad (6)$$

where

$$c_n^{(s)} = \frac{s!}{s+1}(n!)^{-1/2}(-i)^n\sum_{k=0}^{s}\exp(ix_k|\alpha|) \times \mathrm{He}_n(x_k)\mathrm{He}_s^{-2}(x_k). \quad (7)$$

The factors $\mathrm{He}_n(x_k)$ are the modified Hermite polynomials, and x_k are the roots of the Hermite polynomial of order $(s + 1)$: $(\mathrm{He}_{(s+1)}(x_k) = 0)$. The modified Hermite polynomials $\mathrm{He}_n(x)$ are related to the Hermite polynomials H_n by

$$\mathrm{He}_n(x) = 2^{-n/2}\mathrm{H}_n(x/2^{1/2}). \quad (8)$$

3. QUASI-PERIODIC AND PERIODIC BEHAVIOR OF THE FDHS COHERENT STATES

Since in this paper we deal with problems of the quasi-periodicity and periodicity for the FDHS coherent states, we should clarify what kind of periodicity we have in mind. Exploiting the relation between the FDGCS and the state of the field in a cavity with the nonlinear medium discussed earlier, we can treat the FDHS as the space in which the state evolves, where the quantity α plays the role of scaled time ϵt. In consequence, we can study the evolution of the states defined in FDHS as a function of the scaled time α. Hence, we will use the two quantities α and t equivalently. It is seen from the definitions of the expansion coefficients $b^{(s)}$ and $c^{(s)}$ given by (2) and (5) that periodic functions of α appear in the FDGCS coefficient $c^{(s)}$. The coefficients $b^{(s)}$ do not exhibit such periodic properties. Therefore, we will focus our attention mainly on the FDGCS.

As it is seen from (4) and (5), FDGCS can be expressed as a sum of $\cos(x_k\alpha)/x_k^l$ and $\sin(x_k\alpha)/x_k^l$, where $(l = 1, 2)$, and the factor x_k is a root of the Hermite polynomial He. Due to symmetry, we can take into account positive x_k only, i.e., $0 < x_1 < x_2 < \dots$. We are searching for the period $T = \alpha$ fulfilling the relation

$$T = \frac{2\pi}{x_k}(k = 1, 2, \dots). \quad (9)$$

Due to the properties of the roots of the Hermite polynomials, we can write

$$T \simeq \frac{2k\pi}{x_k}. \quad (10)$$

Using the appropriate approximation for the roots of the Hermite polynomials we obtain the following formula for the period T:

$$T \simeq \sqrt{4s+6}. \quad (11)$$

As we will show, the value of T obtained from (9) is particularly close to the exact value of the period for even s.

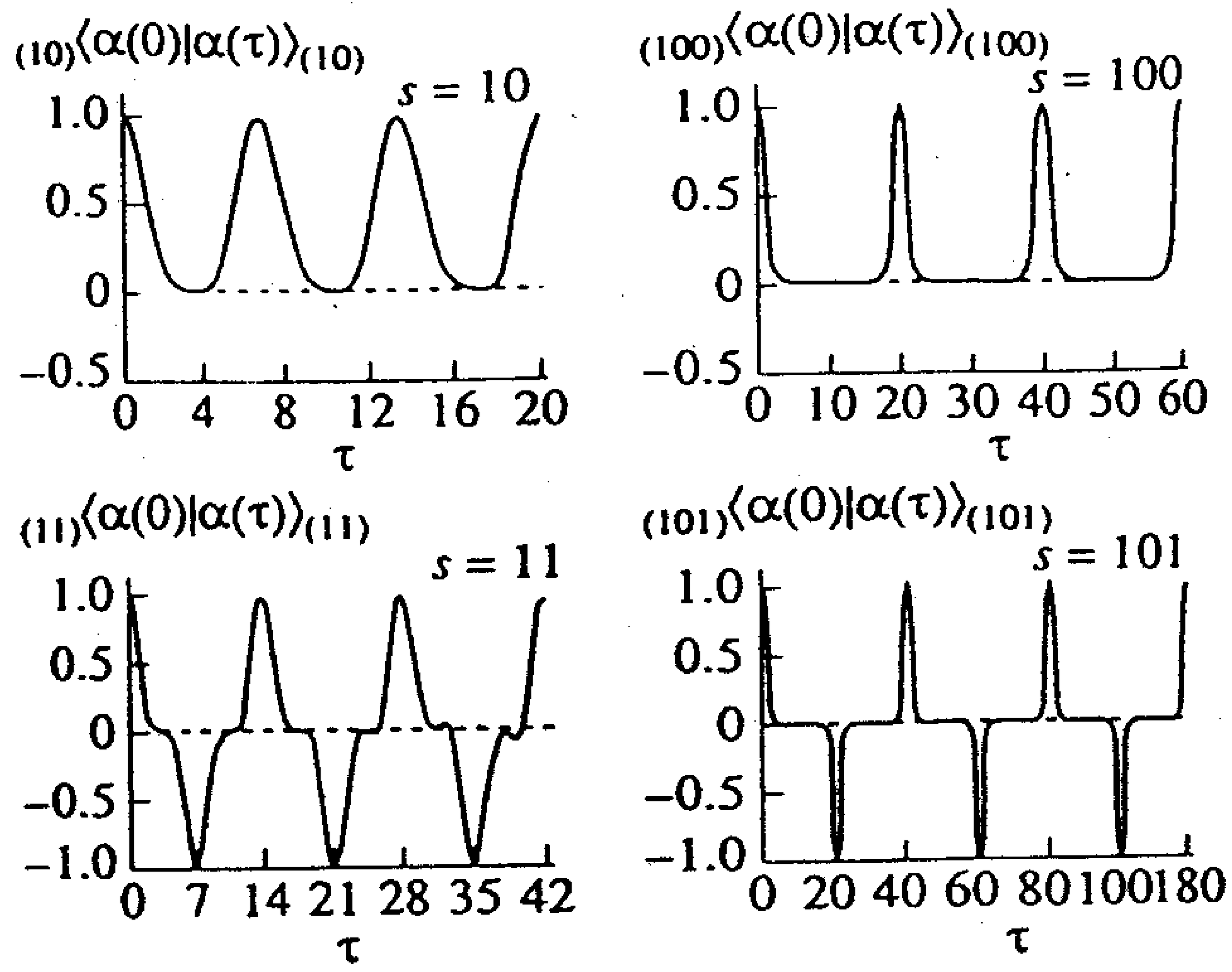

Fig. 1. The scalar product $_{(s)}\langle\alpha(0)|\alpha(\tau)\rangle_{(s)}$ for s = 18, 19, 100, 101: (dotted line) TCS, (solid line) FDGCS.

To show the occurrence of the periodicity in the system (α is treated as a scaled time), we plot the product of two FDHS coherent states $_{(s)}\langle\alpha(t_1)|\alpha(t_2)\rangle_{(s)}$. Assuming that $\alpha \in \Re$ this product becomes

$$_{(s)}\langle\alpha(t_1)|\alpha(t_2)\rangle_{(s)} = f(\alpha(t_2) - \alpha(t_1)) = f(\tau). \quad (12)$$

We plot in Fig. 1 this product as a function of the time τ, assuming that $\alpha(t_1) = 0$. We see that for short times both FDGCS and TCS behave almost identically. For longer times, however, a significant difference between the two states occurs. TCS asymptotically falls down to zero, whereas FDGCS exhibits quasi-periodic behavior. Moreover, we see that for higher dimensions $(s + 1)$ of the space this quasi-periodicity becomes almost ideally periodic. Figure 1 shows the "time evolution" of the product (12) for both even and odd values of s. It is seen that, for odd s, the periodicity is weaker—after each quasi-period additional, nonperiodic features become more and more visible. Nevertheless, they are negligible for higher values of s. In addition, for odd s, the value of the quasi-period T is approximately two times greater than that for its even counterpart. This fact originates from the phase reversal effect. For this case, the value of the scalar product reaches -1. We see that the product reaches the same value as for the initial time but with the opposite sign, contrary to even s, where the product reaches approximately the same value and sign as for the initial time. Moreover, for some values of τ, the scalar product is equal to zero, so the initial state and the state for the time τ are orthogonal.

To show the behavior of the FDGCS during the "time evolution," we plot in (Fig. 2) the distribution of the probability amplitudes $c_n^{(s)}$ for such states with various values of α (treated as the time τ), and for $s = 50$. The evolution starts from the vacuum state $|0\rangle$, and, as the value of α increases, we observe the wave-packet style distribution moving towards higher values of n. As this "packet" reaches the border of the space, interference effects occur. This is an effect of the reflection of the "packet" from the border of the space. After this reflection, the packet moves towards $n = 0$. During this

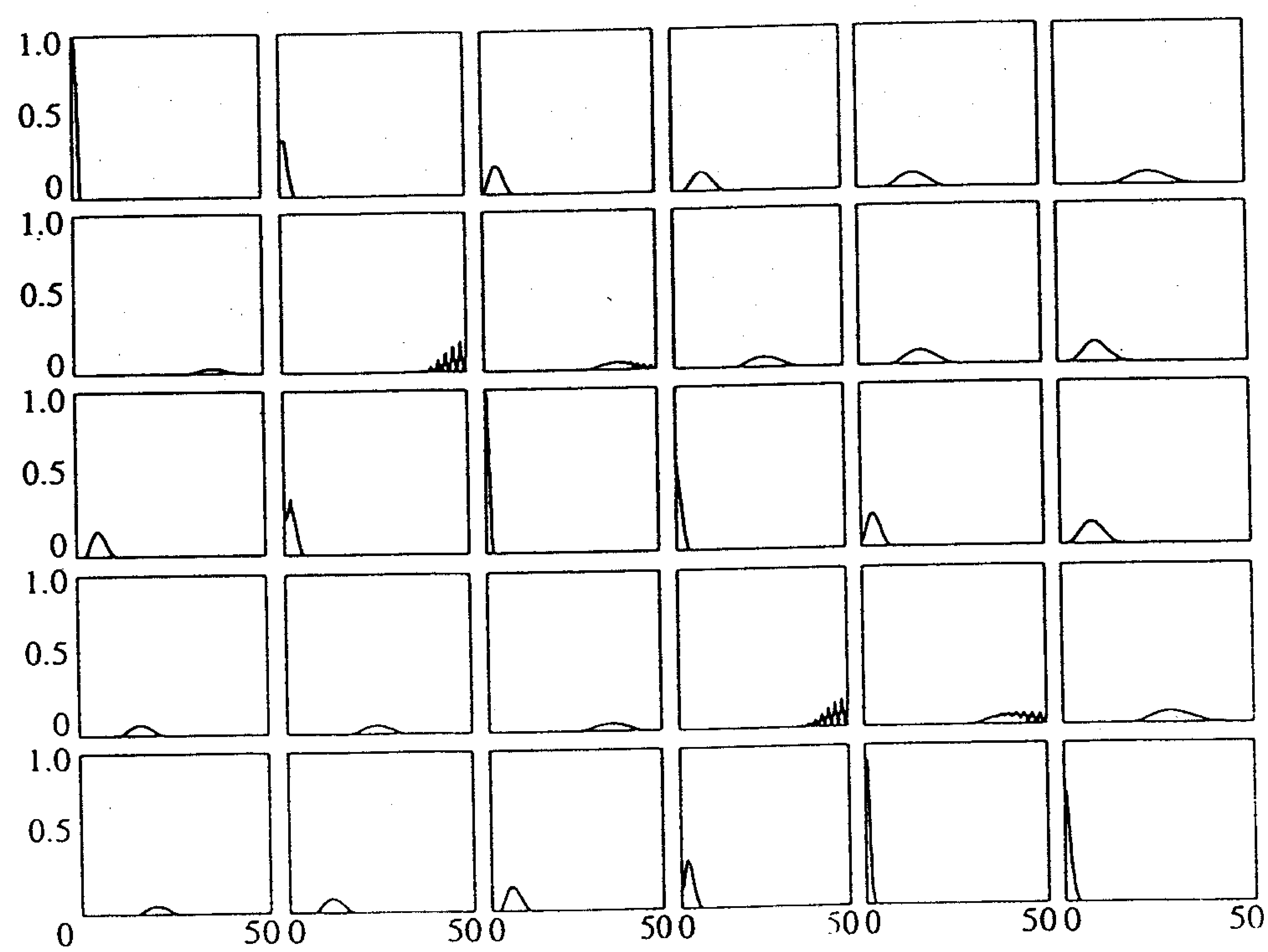

Fig. 2. Distribution of the probability amplitudes $c_n^{(s)}$ for various values of $\tau = \alpha$ (α = 0, ..., 30 with the step $\Delta\alpha$ = 1) and s = 50.

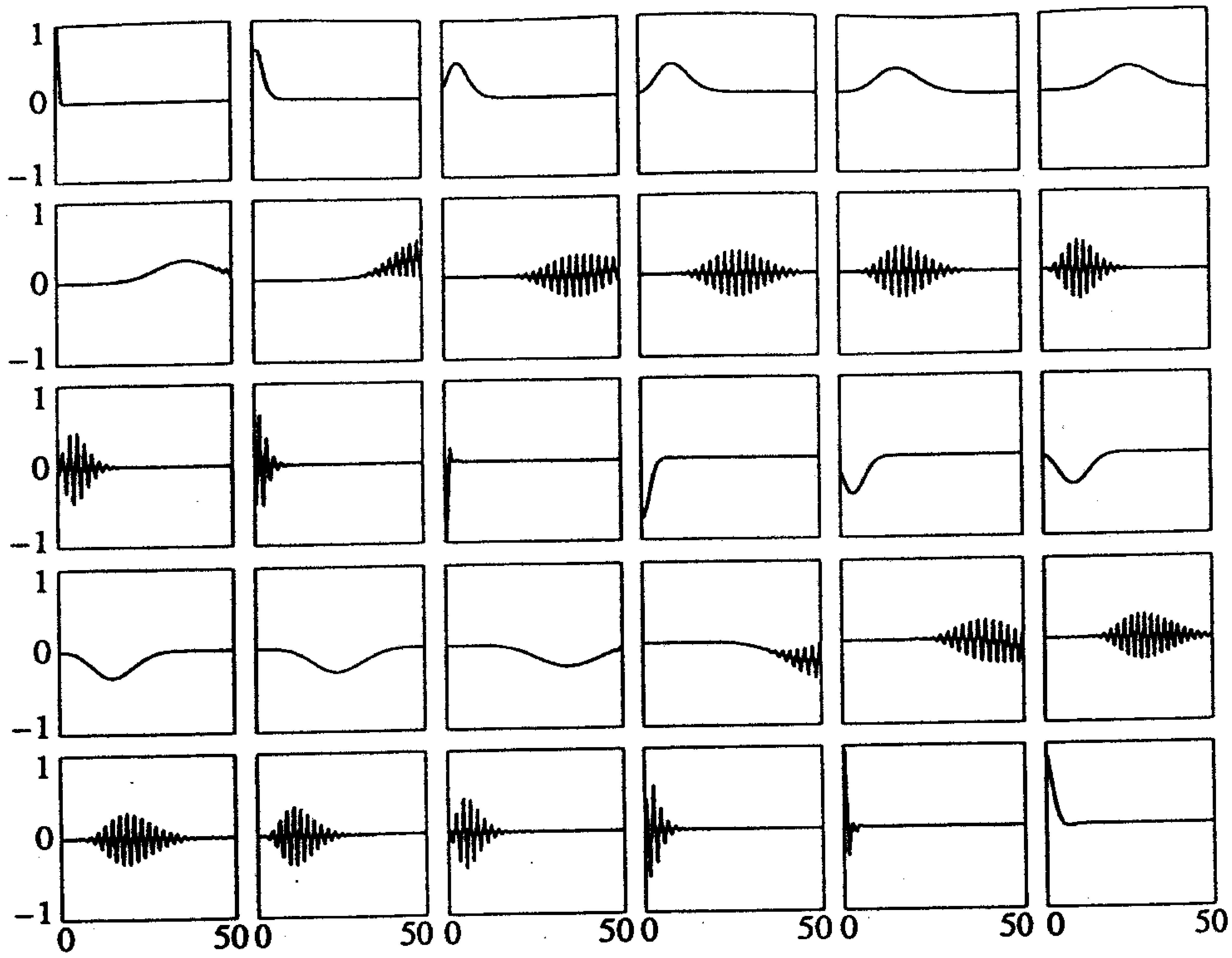

Fig. 3. The same as for Fig. 1 but for $s = 51$.

motion, it changes its shape and becomes a vacuum state distribution again, as for $\tau = 0$. This effect can be referred to as ping effect. For further times, we observe the same behavior as for $\tau < T$ (for $s = 50$ the periodicity is nearly perfect).

Figure 3 shows the same situation but for an odd value of s ($s = 51$). It is seen that for short times the system behaves identically as for the case of $s = 50$. However, as the distribution reaches the second border of the space it changes its sign. This is the same phase reversal effect as for the situation shown in Fig. 1. As a consequence, the interference effect appearing at the space border remains visible during the way of the distribution towards the $n = 0$ state. Moreover, the period T becomes approximately two times greater than that for the case of the even value of s. It is seen that the behavior of the probability distribution depends strongly not only on the value of s, but also on its parity.

Figure 4 shows the function $|_{(s)}\langle\alpha(t)|\alpha(t + T)\rangle_{(s)}|^2$ plotted for various values of s. We have presented plots corresponding to various methods of finding the period T. We compare the results for T found numerically to those derived from our approximate formula (9). We see that for $s = 1, 2$ the FDGCS exhibit perfectly periodic behavior. Nevertheless, as the dimension of the space increases, this periodicity changes to quasi-periodicity. This is a result of the form of the Fock states expansion of the FDGCS. The coefficients $c_n^{(s)}$ are defined by the sum of periodic functions. The "frequencies" inside these functions are determined by the roots of the Hermite polynomials. For $s > 2$, the "frequencies" are not identical and their quotient is not a rational number. In consequence, periodicity is lost. Nevertheless, as s increases, the quasi-periodic behavior tends to be ideally periodic. Moreover, Fig. 4 shows that for even s we achieve better agreement between the approximate solution (9) and T obtained from the numerical calculations. This fact agrees with our earlier discussion concerning derivation of the approximate formula for T.

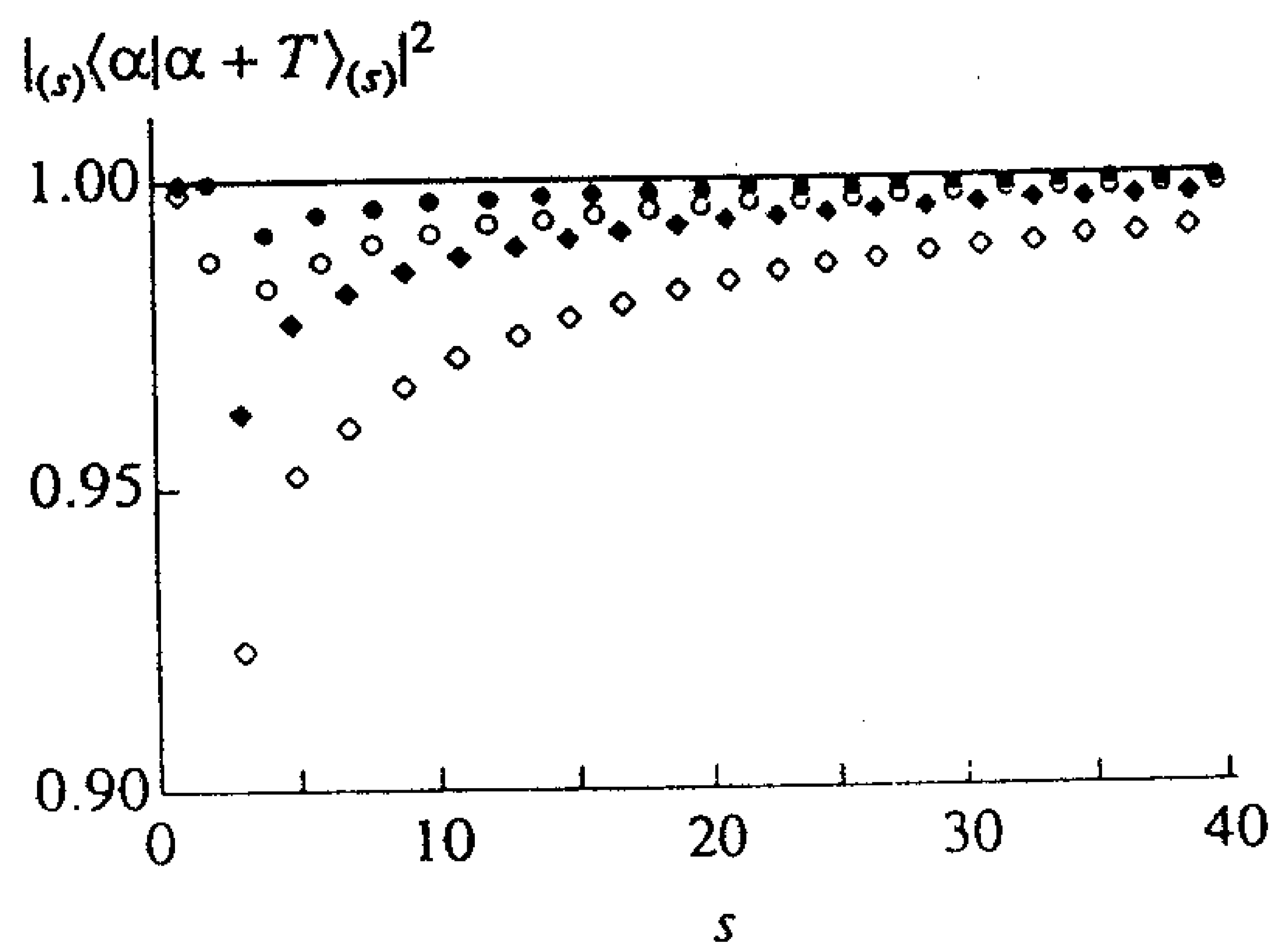

Fig. 4. The quantity $|_{(s)}\langle\alpha|\alpha(\tau + T)\rangle_{(s)}|^2$ for various values of s. Circles correspond to even values of s; square marks, to odd values of s; empty marks, to approximate values of T; and filled marks, to T found numerically.

4. FINAL REMARKS

We have discussed the properties of the finite-dimensional Glauber coherent states and the truncated coherent states. We have paid attention to the quasi-periodic and periodic properties of those states. We have shown that only the FDGCS exhibit such properties. Moreover, the behavior of this kind of states depends on the dimension of the space. For $s = 1, 2$ the states are periodic, whereas for higher values of s they are quasi-periodic. Nevertheless, as s increases, their behavior tends to be periodic. This tendency is more prominent for even values of s. Additionally, we have shown that for odd s the phase reversal effect occurs. We have also shown that the dynamics of the FDGCS expressed in a Fock-state basis exhibits behavior that we have referred to as the ping effect. We have derived an approximate analytical formula for the period T of the quasi-periodic behavior of the system. Our formula is particularly well suited for even s, and agreement between the analytical result and the numerical result becomes better as the value of s becomes greater and greater.

5. ACKNOWLEDGMENTS

This work was carried out within the KBN project nos. 2 P03B 188 8 and 2 P03B 128 8.

REFERENCES

1. Glauber, R.J., 1963, *Phys. Rev. A*, **131**, 2766.
2. Barnett, S.M. and Barnett, P., 1987, *J. Mod. Opt.*, **36**, 7
3. Pegg, D. and Barnett, S.M., 1988, *Europhys. Lett.*, **6** 483.
4. Pegg, D. and Barnett, S.M., 1989, *Phys. Rev. A*, **41**, 3427.
5. Bužek, V., Wilson-Gordon, A.D., Knight, P.L., and Lai, W.K., 1992, *Phys. Rev. A*, 45, 8079.
6. Miranowicz, A., Piątek, K., and Tanaś, R., 1994, *Phys. Rev. A*, **50**, 3423.
7. Kuang, L.M., Wang, F.B., and Zhou, Y.G., 1993, *J. Mod. Opt.*, **42**, 1307.
8. Kuang, L.M., Wang, F.B., and Zhou, Y.G., 1994, *Phys. Lett.*, **183**, 1.
9. Opatrný, T., Miranowicz, A., and Bajer, J., 1996, *J. Mod. Opt.*, **43**, 417.
10. Leoński, W., 1995, *Acta Phys. Slov.*, **45**, 383.