FINITE-DIMENSIONAL SQUEEZED VACUUM AND ITS GENERATION[1]

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Abstract

A squeezed vacuum in a finite-dimensional Hilbert space (FDHS) is defined. Its explicit Fock expansion is found. We show that our state goes over into the standard squeezed vacuum if the squeeze parameter squared is much less than the dimension of FDHS. A method to generate finite-dimensional squeezed vacuum is proposed.

1 Introduction

Various states have recently been constructed in Hilbert spaces with finite number of dimensions. In particular, the following finite-dimensional generalized harmonic-oscillator states have been defined in analogy to the standard (i.e., infinite-dimensional) harmonic oscillator states: (i) various kinds of finite-dimensional coherent states [1]–[4] (ii) finite-dimensional displaced number states [6], (iii) finite-dimensional Schrdinger male and female cats [5, 6], (iv) finite-dimensional phase states [7], (v) finite-dimensional phase coherent states (coherent phase states) [8, 9], and (vi) finite-dimensional displaced phase states [9].

Among the most commonly used states in quantum optics are squeezed states. It is surprising that, as far as we know, there were no proposals of finite-dimensional analogues of the standard squeezed vacuum or the Yuen and Caves squeezed states. In this communication, we shall define squeezed vacuum in FDHS and describe how to generate this state.

Many schemes of Fock state generation have been proposed recently (see, e.g., [10] and references therein) giving the opportunity to construct arbitrary finite superpositions of Fock states. Hence, the methods described in Refs. [10] can be applied to generate the generalized harmonic oscillator states in FDHS, in particular, to construct the squeezed vacuum in FDHS. However in none of these methods, the FDHS states can be generated directly with the proper Fock expansion coefficients. Recently, two of us in Ref. [11], have developed a method of Fock state generation which enables a direct engineering of the FDHS coherent states [12, 13]. Here, we show that this method can be generalized for squeezed vacuum engineering in FDHS.

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2 Definition of squeezed vacuum in FDHS

We are interested in states constructed in the finite-dimensional Hilbert space (FDHS) of a harmonic oscillator. This space, denoted by $\mathcal{H}^{(s)}$, is spanned by $(s+1)$ Fock states which are complete, and orthogonal

$$\widehat{1} = \sum_{n=0}^{s} |n\rangle\langle n|, \quad \langle n|m\rangle = \delta_{n,m}. \tag{1}$$

By analogy with a standard (i.e., infinite-dimensional) squeezed vacuum, we define squeezed vacuum $|\zeta\rangle_{(s)}$ in the $(s+1)$-dimensional Hilbert space by the action of the generalized finite-dimensional squeeze operator

$$\widehat{S}^{(s)}(\zeta) = \exp\left\{\frac{\zeta}{2}(\widehat{a}^{(s)})^{\dagger 2} - \frac{\zeta^*}{2}(\widehat{a}^{(s)})^2\right\} \tag{2}$$

on vacuum, i.e.,

$$|\zeta\rangle_{(s)} = \widehat{S}^{(s)}(\zeta)\,|0\rangle, \tag{3}$$

where $\zeta = |\zeta|\exp(i\varphi)$ is the complex squeeze parameter. The generalized annihilation operator in Eq. (2) is given by

$$\widehat{a}^{(s)} = |0\rangle\langle 1| + \sqrt{2}|1\rangle\langle 2| + \cdots + \sqrt{s}|s-1\rangle\langle s| \tag{4}$$

and the creation operator $(\widehat{a}^{(s)})^\dagger$ is the Hermitian conjugate of $\widehat{a}^{(s)}$. The finite- and infinite-dimensional annihilation operators act on a number state in the same manner. However, the action of the creation operators on $|n\rangle$ is different in $\mathcal{H}^{(s)}$ and $\mathcal{H}^{(\infty)}$, since $(\widehat{a}^{(s)})^{\dagger k}|n\rangle$ is zero whenever $n+k > s$, contrary to the action of the standard creation operator $\widehat{a}^\dagger$. The commutation relation for the annihilation and creation operators in $\mathcal{H}^{(s)}$ reads as

$$[\widehat{a}^{(s)}, (\widehat{a}^{(s)})^\dagger] = 1 - (s+1)|s\rangle\langle s|, \tag{5}$$

which differs from the standard boson canonical relation in $\mathcal{H}^{(\infty)}$, so $\widehat{a}^{(s)}$ and $(\widehat{a}^{(s)})^\dagger$ are not related to the Weyl-Heisenberg algebra. Even the double commutators $[(\widehat{a}^{(s)}), [\widehat{a}^{(s)}, (\widehat{a}^{(s)})^\dagger]]$ and $[(\widehat{a}^{(s)})^\dagger, [\widehat{a}^{(s)}, (\widehat{a}^{(s)})^\dagger]]$ do not vanish precluding the application of the Baker-Hausdorff theorem. These properties of the finite-dimensional annihilation and creation operators considerably complicate the analytical approach to the quantum mechanics in $\mathcal{H}^{(s)}$, including the explicit construction of finite-dimensional generalized harmonic oscillator states.

A method developed by us for the analysis of coherent states in FDHS [4] can be applied here. After some tedious calculations, we find the following explicit Fock expansion of the finite-dimensional squeezed vacuum

$$|\zeta\rangle_{(s)} = \sum_{n=0}^{\sigma} b_{2n}^{(s)} \mathrm{e}^{in\varphi}|2n\rangle, \tag{6}$$

where the superposition coefficients are

$$b_{2n}^{(s)} = (-i)^n \frac{(2\sigma)!}{\sqrt{(2n)!}} \sum_{k=0}^{\sigma} \exp\left(\frac{\mathrm{i}}{2}|\zeta| x_k\right) \frac{G_n(x_k)}{G_\sigma(x_k) G'_{\sigma+1}(x_k)} \tag{7}$$

Here, $\sigma = [\![s/2]\!]$ and the double square bracket denotes the Entier function; $G_n(x)$ are the Meixner-Sheffer orthogonal polynomials [14] defined by the recurrence relation

$$G_{n+1} = xG_n - 2n(2n-1)G_{n-1} \qquad (n = 2, 3, ...) \tag{8}$$

together with $G_0(x) = 1$ and $G_1(x) = x$. In Eq. (7), $x_k \equiv x_k^{(\sigma+1)}$ is the kth root ($k = 0, ..., \sigma$) of the polynomial $G_{\sigma+1}(x)$ and $G'_{\sigma+1}(x_k)$ denotes the x-derivative at $x = x_k$. Since Eq. (7) is of a rather complicated form, we present two examples of the finite-dimensional squeezed vacuum for small dimensions. For s=2 and 3, we find

$$|\zeta\rangle_{(2)} = |\zeta\rangle_{(3)} = \cos\beta|0\rangle + \mathrm{e}^{\mathrm{i}\varphi}\sin\beta|2\rangle, \tag{9}$$

where $\beta = \left|x_0^{(2)}\right| \frac{|\zeta|}{2} = \frac{1}{\sqrt{2}}|\zeta|$. For s=4 and 5, we have

$$|\zeta\rangle_{(4)} = |\zeta\rangle_{(5)} = \frac{1}{7}(6 + \cos\beta)|0\rangle + \ \mathrm{e}^{\mathrm{i}\varphi}\frac{1}{\sqrt{7}}\sin\beta|2\rangle + \ \mathrm{e}^{2\mathrm{i}\varphi}\frac{2\sqrt{6}}{7}\sin^2\left(\frac{\beta}{2}\right)|4\rangle, \tag{10}$$

where $\beta = \left|x_0^{(3)}\right| \frac{|\zeta|}{2} = \sqrt{\frac{7}{2}}|\zeta|$. Our squeezed vacuum (6) has a more complicated form in Fock basis than the finite-dimensional coherent states discussed by us in Ref. [4]. In particular, the solution (6) contains rather complicated Meixner-Sheffer polynomials instead of the well-known Hermite polynomials which occur in the expansions for the FDHS coherent states. A deeper analysis of the properties of squeezed vacuum in FDHS, including photon number and phase properties, squeezed fluctuations and a comparison of the coherent states and squeezed vacua in FDHS, will be presented elsewhere [15]. Here, we discuss only two basic properties of our state. By definition, the squeezed vacuum (3) is properly normalized for arbitrary dimension of the Hilbert space, i.e.,

$$\bigwedge_s \quad {}_{(s)}\langle\zeta|\zeta\rangle_{(s)} = \sum_{n=0}^{\sigma}[b_{2n}^{(s)}]^2 = 1. \tag{11}$$

There are several ways to prove that our finite-dimensional squeezed vacuum $|\zeta\rangle_{(s)}$ goes over into the standard squeezed vacuum $|\zeta\rangle$ in the limit $s \to \infty$. By definition (3), one can conclude that the property $\lim_{s\to\infty}|\zeta\rangle_{(s)} = |\zeta\rangle$ holds, since the finite-dimensional annihilation and creation operators go over into the standard ones, i.e., $\lim_{s\to\infty}(\widehat{a}^{(s)})^{\pm} = \widehat{a}^{\pm}$. One can also show, at least numerically, that the superposition coefficients (7) approach the coefficients b_n for the standard squeezed vacuum, i.e., $\lim_{s\to\infty} b_n^{(s)} = b_n$ for $n = 0, ..., s$. We apply another method based on the calculation of the scalar product $\langle\zeta|\zeta\rangle_{(s)}$. We show the analytical results for $|\zeta| \le 1$ only. The analytical proof for arbitrary $|\zeta|^2 \ll s$ is rather lengthy and therefore will be presented elsewhere [15]. We have found the scalar product between standard ($|\zeta\rangle$) and finite-dimensional ($|\zeta\rangle_{(s)}$) squeezed vacua in the form (for even s)

$$\langle\zeta|\zeta\rangle_{(s)} = \langle\zeta|\zeta\rangle_{(s+1)} = 1 + \sum_{i=1}^{\infty}(-1)^i c_i^{(s)}|\zeta|^{s+2i} \le 1, \tag{12}$$

where the coefficients fulfill the inequalities $0 < c_i^{(s)} < 1$ for any i and s. We find the explicit expression

$$\langle\zeta|\zeta\rangle_{(s)} = \langle\zeta|\zeta\rangle_{(s+1)} = 1 - \binom{s+1}{s/2+1}\left(\frac{|\zeta|}{2}\right)^{s+2} + \mathcal{O}(|\zeta|^{s+4}) \tag{13}$$

in terms of the binomial coefficient. In particular, for s=2,...,5, we have

$$\langle\zeta|\zeta\rangle_{(2)} = \langle\zeta|\zeta\rangle_{(3)} = 1 - \frac{3}{16}|\zeta|^4 + \frac{1}{8}|\zeta|^6 - \frac{167}{2560}|\zeta|^8 + ..., \tag{14}$$

$$\langle\zeta|\zeta\rangle_{(4)} = \langle\zeta|\zeta\rangle_{(5)} = 1 - \frac{5}{32}|\zeta|^6 + \frac{185}{1024}|\zeta|^8 - \frac{419}{3072}|\zeta|^{10} + ... \tag{15}$$

It is clearly seen that, for a given ζ , the scalar product becomes closer to unity with increasing space dimension. This means that the finite-dimensional states $|\zeta\rangle_{(s)}$ approach standard squeezed vacuum.

3 Generation of squeezed vacuum in FDHS

We describe a generation scheme of finite-dimensional squeezed vacuum (3) in a cavity with a nonlinear Kerr medium. The cavity field, which is initially in a vacuum state, is pumped by a train of short pulses (kicks) of the classical electromagnetic field at the frequency of the cavity field. The process is governed by the general time-dependent Hamiltonian

$$\widehat{H}(t) = \widehat{H}_{\text{Kerr}} + \widehat{H}_{\text{kicks}}(t) \tag{16}$$

in the form of an unperturbed system, $\widehat{H}_{\text{Kerr}}$, and a small driving perturbation, $\widehat{H}_{\text{kicks}}(t)$. The unperturbed (between the kicks) evolution of the cavity field in the $(2s+1)$th-order nonlinear Kerr medium [or $(s+1)$-photon anharmonic oscillator] is modelled in the interaction picture by the Hamiltonian [16]:

$$\widehat{H}_{\text{Kerr}} = \frac{\hbar\chi_s}{(s+1)!}(\widehat{a}^\dagger)^{s+1}\widehat{a}^{s+1} \;\equiv\; \frac{\hbar\chi_s}{(s+1)!}\widehat{n}(\widehat{n}-1)\cdots(\widehat{n}-s), \tag{17}$$

where $\widehat{a}$ is the ordinary annihilation operator for the cavity field; $\widehat{n} = \widehat{a}^\dagger\widehat{a}$ is the photon number operator; and χ_s is proportional to the $(2s+1)$th-order nonlinear susceptibility of the medium, $\chi^{(2s+1)}$. The time-dependent Hamiltonian

$$\widehat{H}_{\text{kicks}}(t) = \epsilon\hbar(\widehat{a}^{\dagger 2} + \widehat{a}^2)f(t) \tag{18}$$

describes the second-order parametric process driven by a sequence of short pulses of the classical field. The kick strength ϵ is small enough ($\epsilon < 1$) and therefore will be treated as the strength of perturbation. In general, $f(t)$ is an arbitrary real periodic function of t with the period T. We assume that the time T between the kicks is much longer than $2\pi/\omega$, where ω is the field frequency. Under this assumption, the short pulses of the pump field of frequency ω can be modelled by delta functions, $f(t) = \sum_{m=0}^{\infty}\delta(t - mT)$. If $|\phi(0)\rangle$ is a state at $t = 0$ then the state $|\phi(mT)\rangle$ after m kicks is given by

$$|\phi(t)\rangle = |\phi(mT)\rangle = \widehat{U}^m|\phi(0)\rangle, \tag{19}$$

where the evolution operator $\widehat{U}$ is generated by $\widehat{H}(t)$, which evolves states from $t = 0$ to $t = T$. For $\epsilon = 0$, the system (16) has a simple operator solution [16]. In order to find the eigenstates of $\widehat{U}$ for $0 < \epsilon < 1$, we apply the generalized Rayleigh-Schrdinger perturbation theory, i.e., a generalization of time-independent perturbation theory for systems whose perturbations are in the form of periodic driving [17, 13]. Our problem is equivalent to that of finding the Floquet states, or diagonalizing the sum of the momentum-like operator and the Hamiltonian (16), $-i\hbar\frac{d}{dt}+\widehat{H}(t)$, in the extended Hilbert space $\mathcal{H}\otimes L_2(0,T)$. The degeneracy of the Kerr medium Hamiltonian (17) determines, under certain conditions, the dimension of the Hilbert space $\mathcal{H}^{(s)}$. Finally, after applying the generalized Rayleigh-Schrdinger perturbation theory, we arrive at

$$|\phi(t)\rangle = \sum_{n=0}^{\sigma} C_{2n}^{(s)}(t)|2n\rangle + \epsilon C_{2\sigma+2}^{(s)}(t)|2\sigma+2\rangle + \mathcal{O}(\epsilon^2). \tag{20}$$

The superposition coefficients $C_{2n}^{(s)} = \langle 2n|\phi(t)\rangle_{(s)}$ for $n = 0, ..., \sigma$ are

$$C_{2n}^{(s)}(t) = (-1)^n \frac{(2\sigma)!}{\sqrt{(2n)!}} \sum_{k=0}^{\sigma} \exp(ix_k\epsilon t) \frac{\mathrm{G}_n(x_k)}{\mathrm{G}_\sigma(x_k)\mathrm{G}'_{\sigma+1}(x_k)} \tag{21}$$

and

$$C_{2\sigma+2}^{(s)}(t) = 2^{-\sigma-1}\sqrt{(2\sigma+1)(2\sigma+2)}C_{2\sigma}^{(s)}(t). \tag{22}$$

The same superposition coefficients (10) appear in the Fock expansion of the state (23) for $\zeta = -2i\epsilon t$ if the terms proportional to ϵ are omitted. We finally conclude that

$$|\zeta = -2\mathrm{i}\epsilon t\rangle_{(s)} = |\phi(t)\rangle + \mathcal{O}(\epsilon), \tag{23}$$

meaning that the system described by the effective Hamiltonian (16) evolves into the $(s+1)$-dimensional squeezed vacuum defined by Eq. (3).

4 Conclusions

We have proposed a novel definition of finite-dimensional squeezed vacuum. To our best knowledge, ours is the first definition of a squeezed state analogue in a finite-dimensional Hilbert space. We found an explicit form of the FDHS squeezed vacuum which reveals the differences and similarities between this state and the standard squeezed vacuum or finite-dimensional coherent states. We prove that our state is properly normalized in FDHS for arbitrary dimension and goes over into the standard squeezed vacuum if the dimension is much greater than the square of the squeeze parameter.

We described a physical system comprising a cavity with nonlinear Kerr medium pumped by an external second-order parametric process. By applying the Rayleigh-Schrdinger perturbation theory, we proved analytically that the field generated in this standard model (described in infinite-dimensional Hilbert space) is, under some conditions, finite-dimensional squeezed vacuum. Our generation scheme provides the physical background for the mathematical construction of squeezed vacuum in finite-dimensional Hilbert space.

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