J. Opt. B: Quantum Semiclass. Opt. **1** (1999) 603–609. Printed in the UK PII: S1464-4266(99)05597-4

Comparative study of photon bunching of classical fields

A Miranowicz†¶, H Matsueda†, J Bajer‡, M R B Wahiddin§ and R Tanaś‖

† Department of Information Science, Kochi University, 2-5-1 Akebono-cho, Kochi 780-8520, Japan
‡ Laboratory of Quantum Optics, Palacký University, 772 07 Olomouc, Czech Republic
§ Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia
‖ Institute of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland

Received 30 June 1999

Abstract. Previous work on the two-time photon-number correlations of quantum non-stationary fields has focused on the demonstration of self-contradictory predictions of the photon antibunching according to different definitions. In this paper we analyse the usefulness of the conventional and generalized definitions of photon bunching in a description of classical non-stationary light. It is proved that the generalized definition applied to classical fields predicts photon bunching but never antibunching. In contrast, the two conventional definitions are shown to improperly classify certain classical non-stationary fields as being antibunched.

Keywords: Photon statistics, photon antibunching, frequency conversion

1. Introduction

It is a well known fact that photon antibunching is one of the manifestations of non-classical properties of light. Photon antibunching cannot be understood within the classical field theory describing light as a wave. However, it can be simply interpreted in particle (photon) models as the rise of the joint probability of two detected particles upon the increase of their time separation τ (see, e.g., [1–3]).

In a previous work [4] we systematically compared conventional with generalized approaches to photon antibunching in quantum non-stationary fields. We showed that different definitions can lead to self-contradictory predictions of photon-number correlations in quantum fields. The purpose of this paper is to draw attention to the more severe inconsistency of the conventional descriptions of correlations in the non-stationary regime. We show that some *classical* non-stationary fields can be classified spuriously as antibunched according to the conventional definitions. We apply the generalized definition, which properly excludes the existence of the photon antibunching in classical non-stationary fields.

The photon antibunching effect of non-stationary optical fields has been a subject of considerable interest, particularly, in comparative studies of the sub-Poissonian photon number statistics and antibunching effects. For instance, Singh [5] studied the photon antibunching in the process of resonance fluorescence from a two-level atom in both steady-state and transient regimes. This is, probably, the first paper explicitly showing that the photon-counting statistics can be either sub- or super-Poissonian, even if the photons always exhibit antibunching. Dung *et al* [6] and Aliskenderov *et al* [7] compared non-stationary-field antibunching with sub-Poissonian photon statistics in the Jaynes–Cummings model of a single two-level atom coupled to a single mode of the cavity radiation field. Feng *et al* [8] studied the photon antibunching effects in the model of non-stationary light propagation through a nonlinear fibre with gain. Kryszewski and Chrostowski [9], and Srinivasan and Udayabaskaran [10] (see [3] for references to other similar studies) predicted the photon antibunching of non-stationary fields in parametric frequency up-conversion with stochastic coupling between the signal and idler modes.

We analyse the frequency conversion model to test the definitions of photon antibunching for initial classical fields. We describe a process of exchanging photons between the signal and idler optical modes of different frequencies by the simple Louisell model [11] with trigonometric solutions. The remarkable property of the model is the classical-like evolution or, explicitly, the conservation of initial quasidistributions along classical trajectories as was predicted by Glauber [12] and Mišta [13] and experimentally observed by Huang and Kumar [14] for initial quantum states. The model analysed by Kryszewski and Chrostowski [9], and Srinivasan and Udayabaskaran [10] is a generalized version of the Louisell model constructed by assuming that the mode coupling is stochastic.

The term classical antibunching is ambiguous since it may refer either to antibunching of classical particles or to

¶ Permanent address: Institute of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland.

1464-4266/99/050603+07$30.00

antibunching of classical fields. The antibunching effect is typical for particle models, so is observable for classical particles (see, e.g., classical molecular models analysed by Kielich [15]). However, it cannot be exhibited by classical fields. In the latter this means the classical antibunching is an artefact only.

In section 2, we give a short account of three criteria of photon antibunching. In section 3, we briefly discuss a P-function criterion of classical field evolution in the parametric frequency converter. The main results of the paper are summarized in section 4, where we show explicitly that some classical non-stationary fields can be falsely classified as antibunched according to the two conventional definitions.

2. Three criteria of photon bunching and antibunching

The central role in definitions of the photon antibunching of a single-mode radiation field is played by the intensity correlation function

$$G^{(2)}(t, t+\tau) = \langle \mathcal{T} : \hat{n}(t)\hat{n}(t+\tau) :\rangle \quad (1)$$

directly related, by Glauber's formula [16], to the joint detection probability $P_2(t, t+\tau)$ of detecting one photon at moment t and another at moment $(t+\tau)$. In equation (1), $\widehat{n}(t)$ denotes the photon-number operator, and the operator products are written in normal order (: :), and in time order ($\mathcal{T}$). Different normalizations of $G^{(2)}(t, t+\tau)$ can be applied in the analysis of photon-number correlations. Here, we analyse the normalized two-time second-order intensity correlation functions defined as

$$g_{\rm I}^{(2)}(t, t+\tau) = \frac{G^{(2)}(t, t+\tau)}{[G^{(1)}(t)]^2} \quad (2)$$

$$g_{\rm II}^{(2)}(t, t+\tau) = \frac{G^{(2)}(t, t+\tau)}{G^{(1)}(t)G^{(1)}(t+\tau)} \quad (3)$$

$$g_{\rm III}^{(2)}(t, t+\tau) \equiv \frac{G^{(2)}(t, t+\tau)}{\sqrt{G^{(2)}(t, t)G^{(2)}(t+\tau, t+\tau)}} \quad (4)$$

where $G^{(1)}(t) = \langle n(t)\rangle = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle$ is the light intensity.

The photon antibunching according to the jth (j = I, II, III) definition occurs if the normalized intensity correlation function $g_j^{(2)}(t, t+\tau)$ increases from its initial value at $\tau = 0$, i.e.

$$\Delta g_j(t, t+\tau) \equiv g_j^{(2)}(t, t+\tau) - g_j^{(2)}(t, t) > 0. \quad (5)$$

The photon bunching occurs for decreasing correlation function $g_j^{(2)}(t, t+\tau)$, whereas unbunching takes place if $g_j^{(2)}(t, t+\tau)$ is locally constant. Alternatively, on the assumption that $g_j(t, t+\tau)$ is a well-behaved function of τ, the photon antibunching according to the jth definition occurs if the lowest-order (say n_0) non-vanishing derivative of $g_j^{(2)}(t, t+\tau)$ (or $\Delta g_j(t, t+\tau)$) is positive at $\tau = 0$, i.e., there exists such $n_0 \geqslant 1$ that

$$\gamma_j(t) \equiv \gamma_j^{(n_0)}(t) = \left.\frac{\partial^{n_0}}{\partial\tau^{n_0}} g_j^{(2)}(t, t+\tau)\right|_{\tau=0} > 0 \quad (6)$$

if the derivatives $(\partial/\partial\tau)^n g_j^{(2)}(t, t+\tau)$ vanish at $\tau = 0$ for $n = 1, \ldots, n_0 - 1$. The field exhibits bunching if the lowest-order non-vanishing derivative, $\gamma_j(t)$, is negative. If the derivatives of all orders vanish, $\gamma_j(t) = 0$, the field is said to be unbunched. In the following sections, we use both parameters $\gamma_j(t)$ and correlation functions $\Delta g_j(t, t+\tau)$ to analyse photon bunching in a frequency conversion model.

Definition III, as formulated by the correlation function $g_{\rm III}^{(2)}(t, t+\tau)$, can be rewritten in terms of the so-called correlation coefficient [17]

$$\overline{g}_{\rm III}^{(2)}(t, t+\tau) \equiv \frac{{\rm Cov}\{\hat{n}(t), \hat{n}(t+\tau)\}}{\sigma\{\hat{n}(t)\}\sigma\{\hat{n}(t+\tau)\}} = \frac{\overline{G}^{(2)}(t, t+\tau)}{\sqrt{\overline{G}^{(2)}(t, t)\overline{G}^{(2)}(t+\tau, t+\tau)}} \quad (7)$$

where $\sigma\{\hat{n}(t)\} = \sqrt{\overline{G}^{(2)}(t, t)}$ is the standard deviation of $\hat{n}(t)$ and the central moment

$$\overline{G}^{(2)}(t, t+\tau) \equiv {\rm Cov}\{\hat{n}(t), \hat{n}(t+\tau)\} = G^{(2)}(t, t+\tau) - G^{(1)}(t)G^{(1)}(t+\tau) \quad (8)$$

is the covariance of $\hat{n}(t)$ and $\hat{n}(t+\tau)$ in time and normal order. The correlation coefficient has well known properties and a simple geometrical interpretation as often discussed in probability theory and mathematical statistics (e.g., see [17]).

Both definition I (see [3] and references therein) and definition II (see, e.g., [2]) have been applied to analyse the photon antibunching of non-stationary light generated in various nonlinear optical processes. In particular, analysis of the photon antibunching of *non-stationary* light has been studied by, e.g., Singh [5] and Feng *et al* [8] with the help of definition I, and by, e.g., Kryszewski and Chrostowski [9], Srinivasan and Udayabaskaran [10], Dung *et al* [6] and Aliskenderov *et al* [7] by applying definition II. Definitions I–III are equivalent for stationary fields, i.e., fields satisfying the property $G^{(2)}(t, t+\tau) = G^{(2)}(\tau)$. However, as we have shown in [4], these definitions can lead to self-contradictory predictions of the photon antibunching effect for non-stationary quantum fields.

The classical Cauchy–Schwarz inequality reads as

$$[G^{(2)}(t, t+\tau)]^2 \leqslant G^{(2)}(t, t)G^{(2)}(t+\tau, t+\tau) \quad (9)$$

for the correlation function (1) or, equivalently, as

$$[\overline{G}^{(2)}(t, t+\tau)]^2 \leqslant \overline{G}^{(2)}(t, t)\overline{G}^{(2)}(t+\tau, t+\tau) \quad (10)$$

for the covariance (8). Inequalities (9) and (10) can be violated by non-classical fields only. All definitions of the photon antibunching effect for stationary fields are based on the Cauchy–Schwarz inequality. However, for non-stationary fields, photon antibunching according to definitions I and II does not imply violation of the Cauchy–Schwarz inequality (9). We give examples of classical non-stationary fields apparently exhibiting the antibunching effect according to definitions I and II. In contrast, photon antibunching according to definition III occurs for quantum fields only, independent of the stationary-field condition. This conclusion is readily obtained by comparing the form of the correlation functions $g_{\rm III}^{(2)}(t, t+\tau)$ or $\overline{g}_{\rm III}^{(2)}(t, t+\tau)$ with the Cauchy–Schwarz inequalities (9) or (10), respectively.

3. Classical evolution of the frequency converter

The criterion used to distinguish between the classical and quantum character of light is usually formulated by the Glauber–Sudarshan P-function, i.e., the weight factor in the coherent-state representation of the density matrix

$$\widehat{\rho} = \int d^2\{\alpha_j\} P(\{\alpha_j\}) |\{\alpha_j\}\rangle\langle\{\alpha_j\}|. \tag{11}$$

The classical state of light is defined (see, e.g., [1, 3]) to be one, in which the P-function is a probability distribution, i.e., is neither negative nor more singular than the Dirac δ-function. Otherwise, the state is non-classical. The compact notation for the multimode field is used in equation (11), as the argument $\{\alpha_j\}$ stands for $(\alpha_1, \alpha_2, \ldots)$.

Glauber [12] proved that an initially coherent field remains coherent during the whole evolution of an n-oscillator system described by the Heisenberg equations of motion in the form

$$\frac{d}{dt}\hat{a}_k(t) = F_k(\{\hat{a}_l(t)\}, t), \qquad k, l = 1, \ldots, n \tag{12}$$

for arbitrary functions F_k of the annihilation operators $\hat{a}_l$ and time. In particular, Glauber's theorem applies to the parametric frequency converter, i.e., a process of exchanging photons between signal (a) and idler (b) optical modes of different frequencies (ω_a and ω_b) as described by the Louisell Hamiltonian [11]

$$\hat{H}_{\rm int} = \hbar\kappa\hat{a}_a\hat{a}_b^\dagger \exp(i\Delta\omega t) + \text{h.c.} \tag{13}$$

where $\Delta\omega = \omega + \omega_b - \omega_a$; $\hat{a}_{a,b}$ are annihilation and $\hat{a}_{a,b}^\dagger$ are creation operators; κ denotes the real coupling constant. For simplicity, we assume the resonance case, $\Delta\omega = 0$. The solutions of the Heisenberg equation of motion for the signal and idler modes are [11]

$$\hat{a}_j(t) = \cos(\kappa t)\,\hat{a}_j - i\sin(\kappa t)\hat{a}_{j'} \tag{14}$$

where $\hat{a}_j \equiv \hat{a}_j(0)$, $j = a, b$ and $j' = b, a$, respectively. Glauber's theorem for the frequency converter (13) can be expressed in terms of the Glauber–Sudarshan two-mode P-function found by Mišta [13] in a compact form as

$$P(\alpha_a, \alpha_b, t) = P\{\alpha_a(-t), \alpha_b(-t), 0\} \tag{15}$$

where $\alpha_j(-t)$ are the solutions† of the classical equations of motion for the frequency converter [11]

$$\alpha_j(t) = \cos(\kappa t)\alpha_j - i\sin(\kappa t)\alpha_{j'}. \tag{16}$$

The two-mode P-function remains constant along classical trajectories $\alpha_j(t)$. If both the signal and idler modes are initially classical (non-classical) they will preserve their original character for the whole evolution. Thus, our previous analysis [4] of the photon antibunching in initially Fock states was limited to quantum non-stationary fields. In the present paper, we restrict our photon-correlation analysis to classical non-stationary fields. The Glauber theorem was graphically represented with the help of the Husimi Q-function for various initial statistics in [18]. Preservation of the quantum state during the frequency conversion was experimentally confirmed by Huang and Kumar [14].

† Precisely, the functions $\alpha_j(-t)$ are inverse to the solutions (16). But the inversion is obtained simply by changing the sign of t.

4. Classical photon antibunching artefacts

In a previous work [4] we showed explicitly that the photon-antibunching definitions I–III are not equivalent for quantum non-stationary fields. Nevertheless, the question remains, which of the definitions I–III gives the best indicator of photon antibunching? Here, we analyse their usefulness in a description of photon bunching of classical non-stationary fields.

For simplicity, let us refer to antibunching I, bunching I or unbunching I as to the effects according to definition I. Analogously, we use terms anti-, un- and bunching II or III. We focus our analysis on photon correlations of the signal mode only. Therefore, it will cause no confusion if we omit subscript a in the correlation functions $G^{(2)}(t_1, t_2) \equiv G_a^{(2)}(t_1, t_2)$, $g_j^{(2)} \equiv g_{j,a}^{(2)}$ and $\Delta g_j \equiv \Delta g_{j,a}$.

4.1. Classical antibunching I artefact versus unbunching II and III

If the signal and idler modes are initially coherent, they will remain coherent during the whole evolution of the frequency converter (13). The P-function, according to Glauber's theorem (15), evolves in a classical way

$$\begin{aligned} P(\alpha_a, \alpha_b, t) &= \prod_{j=a,b} \delta(\alpha_j(-t) - \alpha_{j0}) \\ &= \prod_{j=a,b} \delta(\alpha_j - \alpha_{j0}(t)) \end{aligned} \tag{17}$$

where α_{a0} and α_{b0} are the initial amplitudes of the signal and idler modes, respectively; $\alpha_j(-t)$ are the classical solutions (16) for $(-t)$; and $\alpha_{j0}(t) = \cos(\kappa t)\alpha_{j0} - i\sin(\kappa t)\alpha_{j'0}$ for $j = a, b$ and $j' = b, a$, respectively. For simplicity, we analyse the evolution of coherent states with initially real amplitudes α_{a0} and α_{b0}. The unnormalized two-time correlation function (1) for the signal mode is

$$G^{(2)}(t_1, t_2) = \langle n_a(t_1)\rangle\langle n_a(t_2)\rangle \tag{18}$$

as a product of the signal-mode mean intensities

$$\langle n_a(t)\rangle = \alpha_{a0}^2\cos^2(\kappa t) + \alpha_{b0}^2\sin^2(\kappa t) \tag{19}$$

at two evolution times. Thus, the normalized correlation functions are

$$g_{\rm I}^{(2)}(t_1, t_2) = \frac{\langle n_a(t_2)\rangle}{\langle n_a(t_1)\rangle} \tag{20}$$

$$g_{\rm II}^{(2)}(t_1, t_2) = g_{\rm III}^{(2)}(t_1, t_2) = \text{const} = 1. \tag{21}$$

Definitions II and III appear to be equally good since both imply that the coherent field is unbunched

$$\Delta g_{\rm II}(t, t+\tau) = \Delta g_{\rm III}(t, t+\tau) = 0. \tag{22}$$

However, according to definition I, the field might also be bunched or antibunched as seen by expanding solution (20) in a power series of $\tau = t_2 - t_1$ (or $\kappa\tau$). We find

$$\Delta g_{\rm I}(t, t+\tau) = \frac{\alpha_{b0}^2 - \alpha_{a0}^2}{\langle n_a(t)\rangle}\sin(2\kappa t)\,(\kappa\tau) + \mathcal{O}(\tau^2). \tag{23}$$

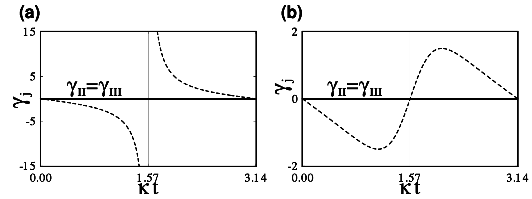

Figure 1. Classical-field evolution of the parameters $\gamma_{\rm I}(t)$ (dashed curves) and $\gamma_{\rm II}(t) = \gamma_{\rm III}(t) = 0$ (solid lines) for the signal and idler fields initially coherent with: (*a*) arbitrary $\alpha_a > 0$ and $\alpha_b = 0$, and (*b*) $\alpha_a = 2$, $\alpha_b = 1$.

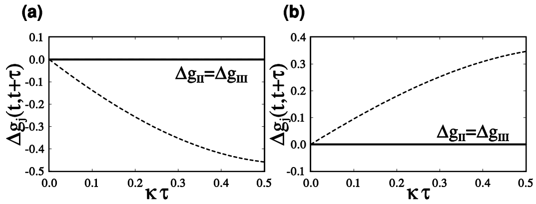

Figure 2. Two-time correlation functions $\Delta g_{\rm I}(t, t+\tau)$ (dashed curves) and $\Delta g_{\rm II}(t, t+\tau) = \Delta g_{\rm III}(t, t+\tau)$ (solid lines) versus rescaled time separation $\kappa\tau$ at fixed evolution times: (*a*) $\kappa t = 1$ and (*b*) $\kappa t = 2.5$ for the same initial condition as in figure 1(*b*).

These counterintuitive outcomes are presented in figures 1 and 2. The time evolution of the parameters γ_j are depicted in figure 1(*a*) for the signal mode initially coherent (with arbitrary non-zero amplitude) and for the idler mode in a vacuum state in figure 1(*a*), or for both fields initially coherent in figure 1(*b*). The exact τ-evolutions of the correlation functions $\Delta g_{\rm I}(t, t+\tau)$ for fixed time t indicate explicitly the bunching effect as presented in figure 2(*a*), but also the classical antibunching artefact according to definition I as depicted in figure 2(*b*).

4.2. Classical antibunching I artefact versus bunching II and III

The classical evolution of the frequency converter with initial chaotic fields is described by the two-mode P-function

$$P(\alpha_a, \alpha_b, t) = \frac{1}{\pi^2} \prod_{j=a,b} \frac{1}{\langle n_{\rm ch,j}\rangle} \exp\left(-\frac{|\alpha_j(-t)|^2}{\langle n_{\rm ch,j}\rangle}\right) \quad (24)$$

where $\langle n_{\rm ch,a}\rangle$ and $\langle n_{\rm ch,b}\rangle$ are the initial mean photon numbers of chaotic photons in the signal and idler modes, respectively. With the help of the relation $\langle (\hat{a}^\dagger)^k \hat{a}^k\rangle = k!\langle n_{\rm ch}\rangle^k$ applied to definition (1), we find

$$G^{(2)}(t_1, t_2) = 2\langle n_{\rm ch,a}\rangle^2 \cos^2(\kappa t_1)\cos^2(\kappa t_2) + 2\langle n_{\rm ch,b}\rangle^2 \sin^2(\kappa t_1)\sin^2(\kappa t_2) + \langle n_{\rm ch,a}\rangle\langle n_{\rm ch,b}\rangle \sin^2[\kappa(t_1+t_2)]. \quad (25)$$

For no time separation, the correlation function (25) simplifies to

$$G^{(2)}(t, t) = 2\langle n_a(t)\rangle^2 \quad (26)$$

where the mean signal-field intensity is

$$\langle n_a(t)\rangle = \langle n_{\rm ch,a}\rangle \cos^2(\kappa t) + \langle n_{\rm ch,b}\rangle \sin^2(\kappa t). \quad (27)$$

Here, in contrast to the evolution of coherent fields, the correlation functions $g^{(2)}_{\rm II}$ and $g^{(2)}_{\rm III}$ are neither the same nor

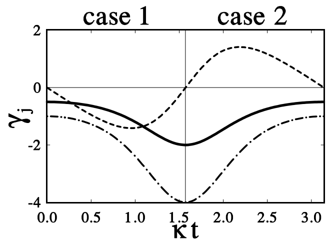

Figure 3. Classical-field evolution of the parameters $\gamma_{\rm I}(t)$ (dashed curve), $\gamma_{\rm II}(t)$ (dot-dashed curve) and $\gamma_{\rm III}(t)$ (solid curve) for the signal and idler fields initially chaotic with the mean photon numbers $\langle n_{\rm ch,a}\rangle = 2$ and $\langle n_{\rm ch,b}\rangle = 1$. Hereafter, the cases (given in the upper part of the figures) correspond to those in table 1.

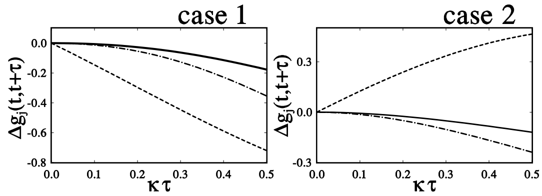

Figure 4. Two-time correlation functions $\Delta g_{\rm I}(t, t+\tau)$ (dashed curves), $\Delta g_{\rm II}(t, t+\tau)$ (dot-dashed curves) and $\Delta g_{\rm III}(t, t+\tau)$ (solid curves) versus rescaled time separation $\kappa\tau$ at $\kappa t = 1$ (case 1) and $\kappa t = 2$ (case 2) for the same initial conditions as in figure 3.

constant. Nevertheless, they are simply related by

$$g^{(2)}_{\rm II}(t_1, t_2) = 2g^{(2)}_{\rm III}(t_1, t_2) \quad (28)$$

as comes from the property (26). The chaotic fields evolving in the frequency converter only exhibit photon bunching according to definitions II and III, as is evident from their power expansions:

$$\Delta g_{\rm II}(t, t+\tau) = 2\Delta g_{\rm III}(t, t+\tau) = -\frac{\langle n_{\rm ch,a}\rangle\langle n_{\rm ch,b}\rangle}{\langle n_a(t)\rangle^2}(\kappa\tau)^2 + \mathcal{O}(\tau^3) < 0 \quad (29)$$

for $\langle n_{\rm ch,a}\rangle$ and $\langle n_{\rm ch,b}\rangle$ different from zero. In contrast, the Taylor expansion

$$\Delta g_{\rm I}(t, t+\tau) = -2\frac{\langle n_{\rm ch,a}\rangle - \langle n_{\rm ch,b}\rangle}{\langle n_a(t)\rangle} \sin(2\kappa t)(\kappa\tau) + \mathcal{O}(\tau^2) \quad (30)$$

implies that the photon antibunching effect in a chaotic field is falsely allowed according to definition I. The evolution of the parameters γ_j, given by the first coefficients of expansions (29) and (30), is presented in figure 3, whereas the correlations $\Delta g_j(t, t+\tau)$, calculated with the help of the exact solutions (25)–(27), are depicted in figure 4. The cases indicated in the upper part of the figures, correspond to those analysed in table 1. It is seen in both figures that the chaotic signal field evolving classically can be bunched (case 1), but also spuriously antibunched (case 2) according to definition I. In contrast, the signal photons can only be bunched according to definitions II and III.

4.3. Classical antibunching I and II artefacts versus bunching III

If the initial modes are in a superposition state of coherent and chaotic fields, the evolution of the frequency converter is

Table 1. All possible predictions of photon antibunching artefacts (described by positive Δg_j) and photon bunching (negative Δg_j) of classical non-stationary fields according to definitions I–III. The initial states $\rho_1(0)$ and $\rho_2(0)$ are given by (39) and (45), respectively.

Case	$\Delta g_{\rm I}$	$\Delta g_{\rm II}$	$\Delta g_{\rm III}$	Examples
1	negative (bunching)	negative (bunching)	negative (bunching)	$\kappa t \in \left(\frac{\pi}{2}, \pi\right)$ for $\rho_2(0)$
2	positive	negative (bunching)	negative (bunching)	$\kappa t \in \left(\frac{\pi}{2}, \pi\right)$ for $\rho_1(0)$
3	negative (bunching)	positive	negative (bunching)	$\kappa t \in \left(0, \frac{\pi}{2}\right)$ for $\rho_1(0)$
4	positive	positive	negative (bunching)	$\kappa t \in \left(0, \frac{\pi}{2}\right)$ for $\rho_2(0)$
5	negative (bunching)	negative (bunching)	positive	forbidden
6	positive	negative (bunching)	positive	forbidden
7	negative (bunching)	positive	positive	forbidden
8	positive	positive	positive	forbidden

described by the P-function

$$P(\alpha_a, \alpha_b, t) = \frac{1}{\pi^2} \prod_{j=a,b} \frac{1}{\langle n_{{\rm ch},j} \rangle} \exp\left(- \frac{|\alpha_j(-t) - \alpha_{j0}|^2}{\langle n_{{\rm ch},j}\rangle} \right). \tag{31}$$

The P-function (31), in the product form of the regular and positive Gaussian functions, explicitly shows that the idler and signal fields remain classical during the frequency conversion. The field evolution described by solution (17), as analysed in section 4.1, and solution (24), as discussed in section 4.2, are the special cases of the evolution described by the P-function (31). Here, we analyse two other special cases.

First, for simplicity, we assume that the mean photon numbers of chaotic photons in both modes are the same $\langle n_{\rm ch,a}\rangle = \langle n_{\rm ch,b}\rangle \equiv \langle n_{\rm ch}\rangle$ and the initial coherent amplitudes α_{j0} are real. By applying relation [19]

$$\langle (\hat{a}^\dagger)^k \hat{a}^k \rangle = k! \langle n_{\rm ch}\rangle^k {\rm L}_k \left\{ - \frac{\beta^2}{\langle n_{\rm ch}\rangle} \right\} \tag{32}$$

where ${\rm L}_k(x)$ is the Laguerre polynomial, we find

$$\begin{aligned} G^{(2)}(t_1, t_2) &= N_+^2 + N_-^2 \cos(2\kappa t_1)\cos(2\kappa t_2) \\ &\quad + 2N_-(N_+ + 2\langle n_{\rm ch}\rangle) \cos[k(t_2 - t_1)] \cos[k(t_1 + t_2)] \\ &\quad + \langle n_{\rm ch}\rangle (2N_+ + \langle n_{\rm ch}\rangle)\{1 + \cos^2[k(t_2 - t_1)]\} \end{aligned} \tag{33}$$

where $N_\pm = \frac{1}{2}(\alpha_{a0}^2 \pm \alpha_{b0}^2)$. The exact expressions for $g_j^{(2)}(t, t+\tau)$ and/or $\Delta g_j(t, t+\tau)$ are calculated from the correlation function (33) by applying the normalization factors

$$\begin{aligned} \langle n_a(t)\rangle &= \langle n_{\rm ch}\rangle + \alpha_{a0}^2 \cos^2(\kappa t) + \alpha_{b0}^2 \sin^2(\kappa t) \\ &= \langle n_{\rm ch}\rangle + \langle n_{\rm coh,a}(t)\rangle \end{aligned} \tag{34}$$

and

$$G^{(2)}(t,t) = \langle n_{\rm coh,a}(t)\rangle^2 + 4\langle n_{\rm ch}\rangle\langle n_{\rm coh,a}(t)\rangle + 2\langle n_{\rm ch}\rangle^2 \tag{35}$$

according to definitions I–III. The mean intensity (34) is the sum of the time-dependent intensity (19) for initially coherent fields and the initial chaotic field intensity. The single-time correlation function (35) is a special case of solution (33). For simpler interpretation, we expand $\Delta g_j(t, t+\tau)$ in a power series of τ arriving at

$$\begin{aligned} \Delta g_{\rm I}(t, t+\tau) &= -2\langle n_a(t)\rangle^{-2}\{\langle n_{\rm ch}\rangle \\ &\quad + \langle n_a(t)\rangle\} N_- \sin(2\kappa t)(\kappa\tau) + \mathcal{O}(\tau^2) \end{aligned} \tag{36}$$

$$\begin{aligned} \Delta g_{\rm II}(t, t+\tau) &= 2\langle n_a(t)\rangle^{-3}\langle n_{\rm ch}\rangle\langle n_{{\rm coh},a}(t)\rangle N_- \sin(2\kappa t)(\kappa\tau) \\ &\quad + \mathcal{O}(\tau^2) \end{aligned} \tag{37}$$

$$\begin{aligned} \Delta g_{\rm III}(t, t+\tau) &= -\frac{\langle n_{\rm ch}\rangle}{[G^{(2)}(t,t)]^2} \{\{G^{(2)}(t,t)(\langle n_{\rm ch}\rangle + 2N_+) \\ &\quad - 4N_-^2\langle n_{\rm ch}\rangle \sin^2(2\kappa t)\}(\kappa\tau)^2 + \mathcal{O}(\tau^3) \leqslant 0. \end{aligned} \tag{38}$$

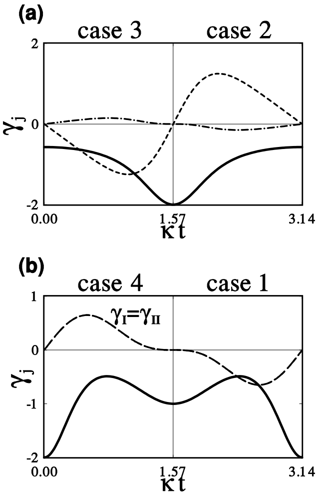

Figure 5. Time evolution of the parameters $\gamma_j(t)$ as in figure 3, but for initial superposition of coherent and chaotic fields: (*a*) $\rho_1(0)$, given by equation (39), and (*b*) $\rho_2(0)$, equation (45).

As for the other fields, the first τ-derivative of $g_{\rm III}^{(2)}(t, t+\tau)$ vanish at $\tau = 0$. Expansions (36)–(38) lead to a simple interpretation. The correlation function $\Delta g_{\rm III}(t, t+\tau)$ cannot be positive, thus we will not observe the antibunching of photons according to definition III. In contrast, both $\Delta g_{\rm I}(t, t+\tau)$ and $\Delta g_{\rm II}(t, t+\tau)$ oscillate between negative and positive values, therefore the antibunching according to definitions I and II is apparently not prohibited. This is our first example of a classical photon antibunching artefact according to definition II, examples of classical antibunching artefacts described by definition I have already been presented in sections 4.1 and 4.2. Surprisingly, the predictions of definitions I and II are opposite, since solutions (36) and (37) have opposite signs and the same time-dependent function. Our conclusion is supported by graphical representations of the parameters γ_j in figure 5(*a*) and $\Delta g_j(t, t+\tau)$ in figure 6 (cases 2 and 3) for the initial condition

$$\rho_1(0) = \rho\{\alpha_a^2 = \langle n_{\rm ch,a}\rangle = \langle n_{\rm ch,b}\rangle = 1, \alpha_b = 0, t = 0\}. \tag{39}$$

Whenever photon bunching is predicted according to either of definitions I or II, it must be accompanied by a classical antibunching artefact according to the other.

As the second example, we analyse another special case of the field (31), in which the evolution is in some sense opposite to the field evolution under the initial condition (39). We assume the signal mode to be initially coherent (with real

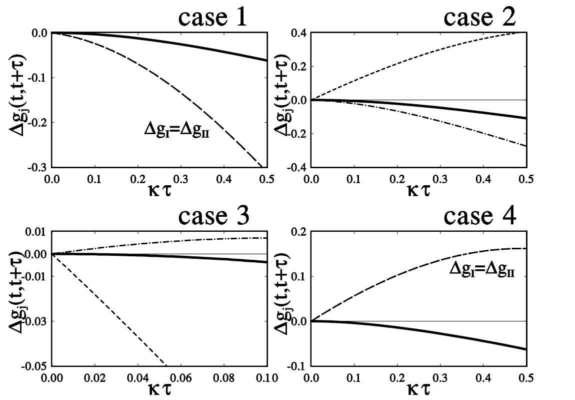

Figure 6. Graphical representation of all four possible different predictions of photon bunching and antibunching artefacts of classical fields as listed in table 1. The two-time signal-mode correlation functions $\Delta g_{\rm I}(t, t+\tau)$ (dashed curves), $\Delta g_{\rm II}(t, t+\tau)$ (dot-dashed curves) and $\Delta g_{\rm III}(t, t+\tau)$ (solid curves) are plotted in their dependence on the rescaled time separation $\kappa\tau$ for fixed values of the evolution time for the initial superpositions of coherent and chaotic fields given by (39) and (45): (case 1) $\rho_2(0)$ at $\kappa t = 2$; (case 2) $\rho_1(0)$ at $\kappa t = 2$; (case 3) $\rho_1(0)$ at $\kappa t = 0.6$, and (case 4) $\rho_2(0)$ at $\kappa t = 0.4$. Evolution times κt were chosen with the help of figure 5.

amplitude α_{a0}), and the idler mode as chaotic (with the mean photon number $\langle n_{\rm ch,b}\rangle$). Then, on applying relation (32), we find the unnormalized correlation function

$$G^{(2)}(t_1, t_2) = \alpha_{a0}^4 \cos^2(\kappa t_1)\cos^2(\kappa t_2) + \alpha_{a0}^2\langle n_{\rm ch,b}\rangle \sin^2[\kappa(t_1+t_2)] + 2\langle n_{\rm ch,b}\rangle^2 \sin^2(\kappa t_1)\sin^2(\kappa t_2) \quad (40)$$

and the mean signal-mode intensity

$$\langle n_a(t)\rangle = \alpha_{a0}^2\cos^2(\kappa t) + \langle n_{\rm ch,b}\rangle \sin^2(\kappa t) \quad (41)$$

required for calculation of the normalized correlations $g_{\rm I}^{(2)}(t, t+\tau)$ and $g_{\rm II}^{(2)}(t, t+\tau)$. The power expansions of the normalized correlations $\Delta g_j(t, t+\tau)$ are

$$\Delta g_{\rm I}(t, t+\tau) = \left(\frac{2[2y\cot(\kappa t) - x\tan(\kappa t)]}{\langle n_a(t)\rangle} - \frac{2xy\tan(\kappa t)}{\langle n_a(t)\rangle^2}\right) \times(\kappa\tau) + \mathcal{O}(\tau^2) \quad (42)$$

$$\Delta g_{\rm II}(t, t+\tau) = \frac{4x^2y\csc(2\kappa t)}{\langle n_a(t)\rangle^3}(\kappa\tau) + \mathcal{O}(\tau^2) \quad (43)$$

$$\Delta g_{\rm III}(t, t+\tau) = -\frac{2\alpha_{a0}^2\langle n_{\rm ch,b}\rangle(x^2+2y^2)}{(x^2+4xy+2y^2)^2} \times(\kappa\tau)^2 + \mathcal{O}(\tau^3) \leqslant 0 \quad (44)$$

where, for brevity, we denote $x = \alpha_{a0}^2\cos^2(\kappa t)$, and $y = \langle n_{\rm ch,b}\rangle\sin^2(\kappa t) = \langle n_a(t)\rangle - x$. The short-time solution (44) reveals the non-positive character of $\Delta g_{\rm III}(t, t+\tau)$, thus excluding the possibility of photon antibunching according to definition III. In contrast, both $\Delta g_{\rm I}(t, t+\tau)$ and $\Delta g_{\rm II}(t, t+\tau)$ change their signs during evolution. On further assumption of equal initial intensities of the signal and idler modes, namely

$$\rho_2(0) = \rho\{\alpha_a^2 = \langle n_{\rm ch,b}\rangle > 0, \alpha_b = \langle n_{\rm ch,a}\rangle = 0, t = 0\}, \quad (45)$$

we find that the normalized correlation functions

$$g_{\rm I}^{(2)}(t_1, t_2) = g_{\rm II}^{(2)}(t_1, t_2) \quad (46)$$

and $g_{\rm III}^{(2)}(t_1, t_2)$ are independent of the initial intensities. Equations (42)–(44) reduce, respectively, to

$$\Delta g_{\rm I}(t, t+\tau) = \Delta g_{\rm II}(t, t+\tau) = \cos^2(\kappa t)\sin(2\kappa t)(\kappa\tau) + \mathcal{O}(\tau^2) \quad (47)$$

$$\Delta g_{\rm III}(t, t+\tau) = -\frac{1+4\sin^2(\kappa t)+3\cos^2(2\kappa t)}{2[2-\cos^4(\kappa t)]^2}(\kappa\tau)^2 + \mathcal{O}(\tau^3) \leqslant 0. \quad (48)$$

Evidently, solution (47) takes positive values at some evolution times. We conclude that the classical antibunching artefact according to definition I occurs whenever it exists according to definition II for the signal under the initial condition (45). These results are graphically represented in figure 5(*b*) and figure 6 (cases 1 and 4). It is worth comparing solution (47) with equations (36) and (37) describing opposite (out-of-phase) behaviour of $\Delta g_{\rm I}(t, t+\tau)$ and $\Delta g_{\rm II}(t, t+\tau)$ (see figure 5(*a*)).

Table 1 summarizes our investigations of photon bunching effects in classical fields. By virtue of the Cauchy–Schwarz inequality, photon antibunching according to definition III cannot occur for classical fields, thus cases 5–7 in table 1 are excluded. However, the remaining cases 1–4 are observed in the evolution of classical fields as presented in figures 5 and 6. The classical photon antibunching apparently exists according to both definitions I and II.

Photon antibunching of classical fields can only be an artefact. So, it seems necessary to modify the conventional definitions in the non-stationary regime. For instance, one can add an extra condition, which guarantees the quantum character of the field but keeps the original inequalities unchanged. Nevertheless, the problem of the unique description of photon antibunching in non-stationary case would remain in the conventional definitions. In contrast, these problems do not arise in the generalized approach to photon antibunching (definition III), where the Cauchy–Schwarz inequality is applied directly without any further assumptions.

5. Conclusions

We have demonstrated that the photon antibunching according to the conventional definitions I and II for nonstationary fields does not imply violation of classical inequalities, including that of Cauchy and Schwarz. Moreover, we have devised classical (as described by regular and positive-definite P-function) nonstationary fields, which fulfil the criteria I and II for photon antibunching.

'Definitions can be neither right nor wrong, and their merit is determined only by their usefulness' [20]. Definitions I and II can still be useful in a description of photon antibunching for not only stationary fields, but also quantum non-stationary fields. However, in the latter case, the non-classical character of the fields should be checked independently, e.g. with the help of the P-function criterion or Cauchy–Schwarz inequality.

In comparison with definitions I and II, we have used the generalized criterion (definition III) to describe photon antibunching of non-stationary fields. These three criteria are close to the original photodetection interpretation for stationary fields [1, 4], but only definition III guarantees that the photon antibunching cannot occur for classical non-stationary light. The generalized definition of photon antibunching is given on the basis of the Cauchy–Schwarz inequality without any assumptions concerning properties of the fields. Whereas the conventional definitions I and II come from the Cauchy–Schwarz inequality under the stationary-field condition. Thus, antibunching according to the generalized definition cannot occur for classical fields. In contrast, as we have shown for the parametric frequency converter with classical initial conditions, the conventional definitions I and II can lead to the photon antibunching artefacts of certain classical non-stationary fields.

Acknowledgments

We thank Marc Dupertuis, Tuğrul Hakioğlu, Maciej Kozierowski and Jan Peřina for their comments. AM gratefully acknowledges the scholarship of the Japanese Ministry of Education. HM was supported within the Proposal-Based New Industry Creative Type Technology R&D Promotion Programme by the New Energy and Industrial Technology Development Organization (NEDO) of Japan. JB was supported by the Czech Ministry of Education (grant no VS96028) and the Czech Republic Grant Agency (no 202/96/0421). MRBW acknowledges support from the Malaysia S&T IRPA (grant no 09-02-03-0337). RT was supported by the Polish Research Committee (grant no 2 PO3B 73 13).

References

[1] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)
[2] Teich M C and Saleh B E A 1988 *Progress in Optics* ed E Wolf (Amsterdam: Elsevier) vol 26, p 1
[3] Peřina J 1991 *Quantum Statistics of Linear and Nonlinear Optical Phenomena* (Dordrecht: Kluwer)
[4] Miranowicz A, Bajer J, Matsueda H, Wahiddin M R B and Tanaś R 1999 *J. Opt. B: Quantum Semiclass. Opt.* **1**
[5] Singh S 1983 *Opt. Commun.* **44** 254
[6] Dung H T, Shumovsky A S and Bogolubov N N Jr 1992 *Opt. Commun.* **90** 322
[7] Aliskenderov E I, Dung H T and Knöll L 1993 *Phys. Rev.* A **48** 1604
[8] Feng L-Y, Qian F and Deng L-B 1994 *J. Mod. Opt.* **41** 431
[9] Kryszewski S and Chrostowski J 1977 *J. Phys. A: Math. Gen.* **10** L261
[10] Srinivasan S K and Udayabaskaran S 1979 *Opt. Acta* **26** 1535
[11] Louisell W 1964 *Radiation and Noise in Quantum Electronics* (New York: McGraw-Hill) p 274
[12] Glauber R J 1966 *Phys. Lett.* **21** 650
[13] Mišta L 1969 *Czech. J. Phys.* B **19** 443
[14] Huang J and Kumar P 1992 *Phys. Rev. Lett.* **68** 2153
[15] Kielich S 1981 *Nonlinear Molecular Optics* (Moscow: Nauka) (in Russian)
[16] Glauber R J 1963 *Phys. Rev.* **130** 2529
[17] Berger M A 1993 *An Introduction to Probability and Stochastic Processes* (New York: Springer)
Sinay Y G 1992 *Probability Theory* (Berlin: Springer)
Shiryayev A N 1984 *Probability* (New York: Springer)
[18] Miranowicz A, Matsueda H, Wahiddin M R B and Bajer J 1998 *Acta Phys. Slov.* **48** 293
[19] Mollow B R and Glauber R J 1967 *Phys. Rev.* **160** 1097
Peřina J and Mišta L 1968 *Czech. J. Phys.* B **18** 697
[20] Senitzky I R 1964 *Quantum Electronics* ed P Grivet and N Bloembergen (Paris: Dunod) p 120