Decoherence of quantum operations: How to coherify a classical map?

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in collaboration with

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V QuTecNOMM 2019,
UAM, Poznań, **November 21, 2019**

Department of Physics, Jagiellonian University, Cracow,

view from my new office,


WYWÓZ NIECZYSTOŚCI
PŁYNNYCH-STAŁYCH-WUKO
270 31 80, 266 04 73
UDRAŻNIANIE KANALIZACJI


SWIETELSKY
FABER
KANALSANIERUNG

Quantum
Kanalsanierung !

Which **quantum channel**
could be called
healthy and **sane** ?

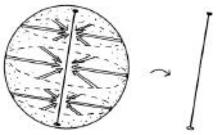
Perhaps
a *unitary* and
reversible one ?

Sanierung of **Quantum States** acting on $\mathcal{H}_N$

Convex set $\mathcal{M}_N \subset \mathbb{R}^{N^2-1}$ of all mixed states of size N

$$\mathcal{M}_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N ; \rho = \rho^\dagger, \rho \geq 0, \mathrm{Tr}\rho = 1\}$$

example: $\mathcal{M}_2 = B_3 \subset \mathbb{R}^3$ - Bloch ball with all pure states at the boundary



Quantum decoherence: pure $\to$ mixed
stripping off–diagonal elements,

$$\mathcal{D}(\rho) = \sum_i \rho_{ii} |i\rangle\langle i| = \mathrm{diag}(\rho)$$

projection into the simplex of classical states

A) Purification of $\rho \in \mathcal{M}_N$

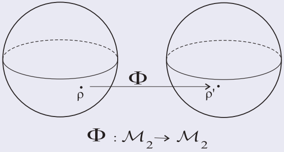
search of a bi-partite pure state $|\psi_{AB}\rangle \in \mathcal{H}_N \otimes \mathcal{H}_N$
such that the reduced matrix reads $\mathrm{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}| = \rho$.

B) Coherification of a classical state $\mathrm{diag}(p) = \sigma \in \mathcal{M}_N$

search of a mono-partite pure state $|\phi_A\rangle \in \mathcal{H}_N$ such that it decohers to the diagonal, classical state, $\mathcal{D}(|\phi_A\rangle\langle\phi_A|) = \sigma = \mathrm{diag}(p)$.

Quantum Channels

Quantum operation: linear, completely positive trace preserving map



positivity: $\Phi(\rho) \geq 0, \quad \forall \rho \in \mathcal{M}_N$

complete positivity: $[\Phi \otimes \mathbb{1}_K](\sigma) \geq 0, \quad \forall \sigma \in \mathcal{M}_{KN}$ and $K = 2, 3, \ldots$

Enviromental form (interacting quantum system !)

$$\rho' = \Phi(\rho) = \mathrm{Tr}_E[U\,(\rho \otimes \omega_E)\,U^\dagger]\ .$$

where ω_E is an initial state of the environment while $UU^\dagger = \mathbb{1}$.

Kraus form

$\rho' = \Phi(\rho) = \sum_i A_i \rho A_i^\dagger, \quad$ where the Kraus operators satisfy $\sum_i A_i^\dagger A_i = \mathbb{1}$, which implies that the trace is preserved.

Classical & Quantum discrete dynamics

Stochastic matrices

Classical states: N-point probability distribution, $\mathbf{p} = \{p_1, \ldots p_N\}$,
where $p_i \geq 0$ and $\sum_{i=1}^{N} p_i = 1$
Discrete dynamics: $p_i' = T_{ij} p_j$, where T is a **stochastic transition matrix** of size N and maps the simplex of classical states into itself,

$$T: \ \Delta_{N-1} \to \Delta_{N-1}.$$

Stochastic maps = quantum operations

A *quantum operation* $\Phi: \ \mathcal{M}_N \to \mathcal{M}_N$
can be described by a matrix Φ of size N^2,

$$\rho' = \Phi\rho \qquad \text{or} \qquad \rho'_{m\mu} = \Phi_{\substack{m\mu \\ n\nu}} \rho_{n\nu} \ .$$

The superoperator Φ can be expressed in terms of the Kraus operators A_i,

$$\Phi = \sum_i A_i \otimes \bar{A}_i \ .$$

Quantum stochastic maps (trace preserving, CP)

Dynamical Matrix D: Sudarshan et al. (1961)

obtained by *reshuffling* of a 4–index matrix Φ is Hermitian,

$$D_{\substack{mn \\ \mu\nu}} := \Phi_{\substack{m\mu \\ n\nu}} \ , \quad \text{so} \quad \text{that} \quad D_\Phi = D_\Phi^\dagger =: \Phi^R \ .$$

Theorem of Choi (1975). A map Φ is **completely positive** (CP) if and only if the dynamical matrix D_Φ is **positive**, $D \geq 0$.

Classical case

In the case of a **diagonal dynamical matrix**, $D_{ij} = d_i \delta_{ij}$ reshaping its diagonal $\{d_i\}$ of length N^2 one obtains a matrix of size N, where $T_{ij} = D_{\substack{ii \\ jj}}$, of size N which is **stochastic**.

Decoherence for quantum states and quantum maps

Quantum states $\to$ classical states $=$ diagonal matrices

Decoherence of a state: $\rho \to \Phi_{\rm CG}(\rho) = \tilde{\rho} = {\rm diag}(\rho)$

Quantum maps $\to$ classical maps $=$ stochastic matrices

(Hyper-) decoherence of a **map**: The **Choi matrix** becomes diagonal, $D \to \Gamma_{\rm CG}(D) = \tilde{D} = {\rm diag}(D)$ so that the map $\Phi = D^R \to \tilde{D}^R \to T$.
For any Kraus decomposition defining $\Phi(\rho) = \sum_i A_i \rho A_i^\dagger$ the corresponding **classical map** T is given by the **Hadamard product**,

$$T = \Gamma_{\rm CG}(\Phi) = \sum_i A_i \odot \bar{A}_i,$$

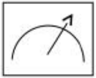
where $\Gamma_{\rm CG}$ is the **coarse–graining supermap**, **K.Ż.** (2008)

If a **quantum map** Φ is trace preserving, $\sum_i A_i^\dagger A_i = \mathbb{1}$
then the **classical map** $T = \Gamma_{\rm CG}(T)$ is **stochastic**, $\sum_j T_{ij} = 1$.

If additionally a **quantum map** Φ is unital, $\sum_i A_i A_i^\dagger = \mathbb{1}$
then the **classical map** T is **bistochastic**, $\sum_j T_{ij} = \sum_i T_{ij} = 1$.

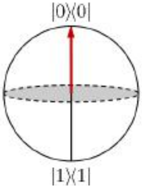
Infering an information on a state and a map

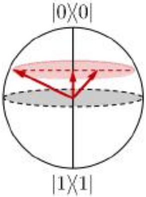
Quantum state ρ

$\rho \longrightarrow$  $\longrightarrow p_j = \langle j|\rho|j\rangle$

What **p** tells us about ρ?

$\mathbf{p} = [1, 0]$ $\qquad$ $\mathbf{p} = [3/4, 1/4]$





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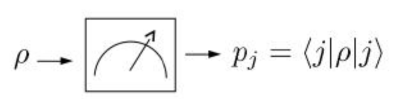

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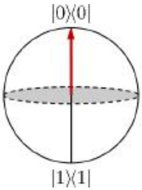

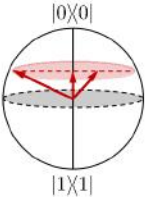

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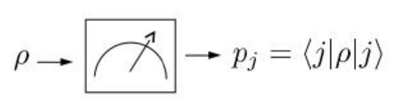

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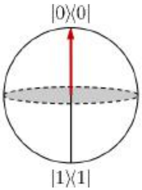
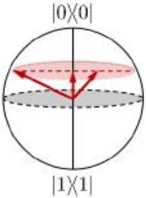
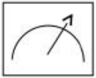




Infering an information on a state and a map

Quantum state ρ

$\rho \rightarrow$  $\rightarrow p_j = \langle j|\rho|j\rangle$

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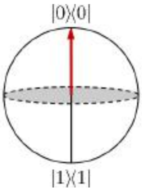

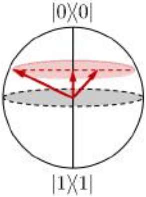

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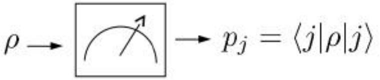

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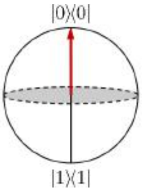

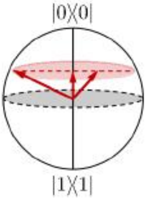

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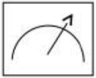
$\rho \longrightarrow$  $\longrightarrow p_j = \langle j|\rho|j\rangle$

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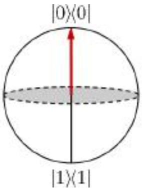

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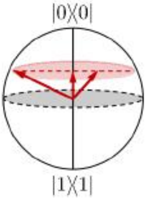

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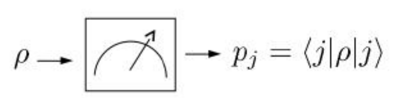

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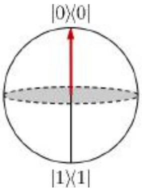
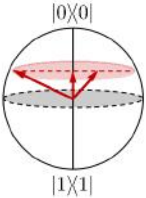
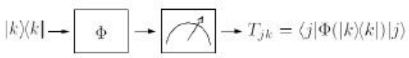




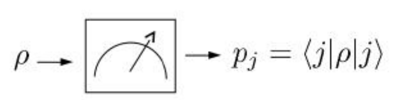
Quantum channel Φ



What T tells us about Φ?

Infering an information on a state and a map

Quantum state ρ



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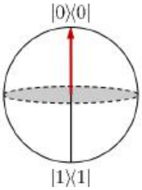
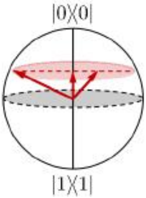
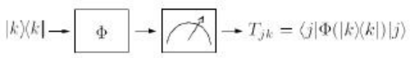


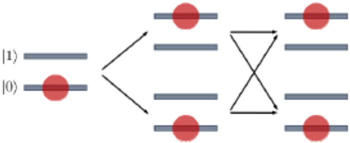


Quantum channel Φ



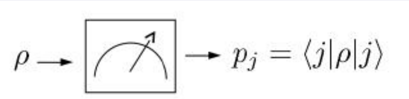
What T tells us about Φ?

$T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, *depolarization* $\Phi_*(\rho) = \frac{1}{2}\mathbb{1}$



Infering an information on a state and a map

Quantum state ρ



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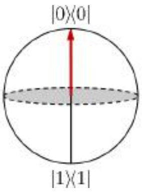

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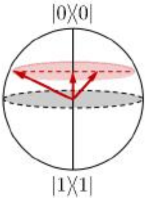

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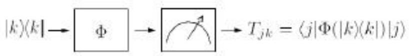
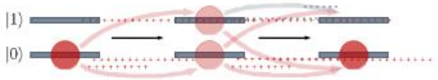


What T tells us about Φ?

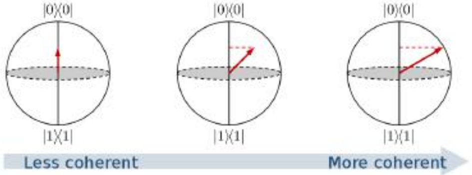
$T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, can describe unitary map

$$\Phi_H(\rho) = H(\rho)H^\dagger, \quad H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$



Coherence of quantum states

Given a fixed basis $\{j\}$
with $j \in \{1, 2, \ldots, N\}$
populations $p_j = \langle j|\rho|j\rangle$:
coherences $c_{jk} = \langle j|\rho|k\rangle$



$$\rho_1 = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \qquad \rho_2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad \rho_3 = \begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

Coherence of quantum states

Decohering channel $\mathcal{D}$

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populations $p_j = \langle j|\rho|j\rangle$:
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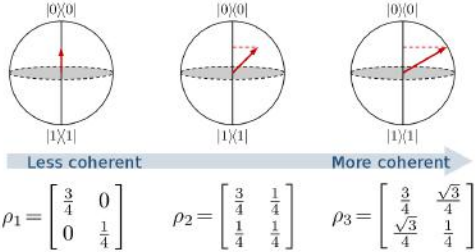
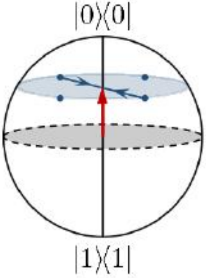
$$\mathcal{D}(\rho) = \sum_{j=1}^{N} \langle j|\rho|j\rangle |j\rangle\langle j|$$





$c_{jk} \to 0,\ p_j \to p_j$

$\mathcal{D}(\rho_1) = \mathcal{D}(\rho_2) = \mathcal{D}(\rho_3) = \rho_1.$

Classical bit embedded inside

$|0\rangle$  $|1\rangle$

Classical bit embedded inside

$|0\rangle$  $|1\rangle$

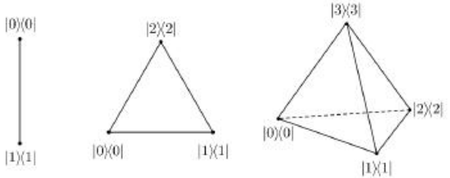
the **Bloch ball** and its ... **decoherence**

Coherence of quantum states

Incoherent state ρ is identified with a **classical** probability distribution p.

$$\rho = \mathcal{D}(\rho) = \sum_{j=1}^{N} p_j |j\rangle\langle j|$$

Classical state space
=
probability simplex Δ_{N-1}



Coherence measures (a *distance* from incoherent states)

$$\textit{entropic}: \quad C_e(\rho) = S(\rho||\mathcal{D}(\rho)) = S(p) - S(\lambda(\rho))$$
$$\textit{geometric}: \quad C_2(\rho) = \|\rho - \mathcal{D}(\rho)\|^2_{HS} = \lambda(\rho)\cdot\lambda(\rho) - p\cdot p$$

Baumgratz, Cramer, Plenio, (2014)
Streltsov, Adesso, Plenio, (2016)

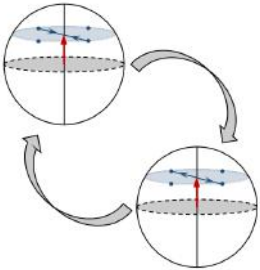
Coherifying quantum states

Decohering channel $\mathcal{D}$:

$$\rho \xmapsto{\mathcal{D}} \rho^{\mathcal{D}} = diag(p)$$

Coherification $\mathcal{C}$ is a formal (not unique!) inverse of $\mathcal{D}$:

$$\rho = diag(p) \xmapsto{\mathcal{C}} \rho^{\mathcal{C}}$$



One can always optimally **coherify** a **classical state** p:

$$\rho = diag(p) \xmapsto{\mathcal{C}} |\psi\rangle\langle\psi| \quad \text{with} \quad |\psi\rangle = \sum_{j=1}^{N} \sqrt{p_j} e^{i\phi_j} |j\rangle$$

Coherifying quantum states

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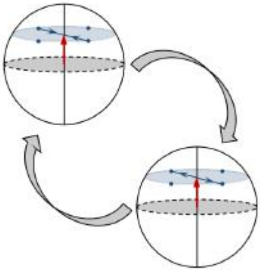

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$$C_e(|\psi\rangle\langle\psi|) = S(p), \quad C_2(|\psi\rangle\langle\psi|) = 1 - p \cdot p.$$

How many distinct ways to **coherify?**

Coherence of quantum channels

Given a fixed basis $\{|j\rangle\}$, with $j \in \{1, 2, \ldots, N\}$

$\langle j|\Phi(|k\rangle\langle k|)|j\rangle$: classical action T_{jk}
$\langle j|\Phi(|m\rangle\langle n|)|k\rangle$:
action involving coherences

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action involving coherences

Choi-Jamiołkowski
isomorphism
channel $\Phi \longleftrightarrow$ bipartite state

$$J_\phi = \frac{1}{N}(\Phi \otimes \mathbb{1})|\Omega\rangle\langle\Omega| \ , \ |\Omega\rangle = \sum_j |jj\rangle$$

CP & trace preserving

conditions are translated into:

$$J_\Phi \geq 0, \quad \mathrm{tr}_1(J_\Phi) = \frac{1}{N}\mathbb{1}$$

Coherence of quantum channels

Given a fixed basis $\{|j\rangle\}$, with $j \in \{1, 2, \ldots, N\}$

$\langle j|\Phi(|k\rangle\langle k|)|j\rangle$: classical action T_{jk}
$\langle j|\Phi(|m\rangle\langle n|)|k\rangle$:
action involving coherences

Choi-Jamiołkowski
isomorphism
channel $\Phi \longleftrightarrow$ bipartite state

$J_\phi = \frac{1}{N}(\Phi \otimes \mathbb{1})|\Omega\rangle\langle\Omega|$, $|\Omega\rangle = \sum_j |jj\rangle$

CP & trace preserving

conditions are translated into:

$J_\Phi \geq 0, \quad \mathrm{tr}_1(J_\Phi) = \frac{1}{N}\mathbb{1}$

Relation between J_Φ and T:

$\langle j,k|J_\Phi|j,k\rangle = \frac{1}{N}T_{jk}$
$diag(J_\Phi) = \frac{1}{N}|T\rangle\rangle$,
matrix T reshaped into a vector

Vectorising classical action:
where $|T\rangle\rangle = T \otimes \mathbb{1}|\Omega\rangle$

Coherence of quantum channels

Classical channels are defined as **channels** with incoherent (**classical**) Jamiołkowski state.

Action of classical channel described by the transition matrix T

$$\rho \mapsto \mathcal{D}(\rho) = \sum_j p_j |j\rangle\langle j| \mapsto \sigma = \sum_j q_j |j\rangle\langle j| \quad \text{with } q = Tp$$

Define **coherence measure** of a map Φ by **coherence measure** of J_Φ

$$C_e(\Phi) = S(\frac{1}{N}|T\rangle\rangle) - S(\lambda(J_\Phi)), \quad C_2(\Phi) = \lambda(J_\Phi)\cdot\lambda(J_\Phi) - \frac{1}{N^2}\langle\langle T||T\rangle\rangle$$

In analogy to:

$$C_e(\rho) = S(\rho||\mathcal{D}(\rho)) = S(p) - S(\lambda(\rho))$$
$$C_2(\rho) = \lambda(\rho)\cdot\lambda(\rho) - p\cdot p$$

Approach differs from *cohering power* of a channel:
Mani, Karimipour, (2015); **Zanardi, Styliaris, Venuti**, (2017)

Coherence of quantum channels

Decohering operation $\mathcal{D}$

Φ with $diag(J_\Phi) = \frac{1}{N}|T\rangle\rangle \mapsto \Phi^{\mathcal{D}}$ with $J_{\Phi^{\mathcal{D}}} = \mathcal{D}(J_\Phi) = \frac{1}{N} diag(|T\rangle\rangle)$

Coherification $\mathcal{C}$ (not unique!) inverse of $\mathcal{D}$

Φ with $J_\Phi = \mathcal{D}(J_\Phi) = \frac{1}{N} diag(|T\rangle\rangle) \mapsto \Phi^{\mathcal{C}}$ with $diag(J_{\Phi^{\mathcal{C}}}) = \frac{1}{N}|T\rangle\rangle$

Can one always optimally coherify a classical map T?

Coherence of quantum channels

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Can one always optimally coherify a classical map T?

$\frac{1}{N}|T\rangle\rangle \mapsto |\psi\rangle\langle\psi|$ with

$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{jk} \sqrt{T_{jk}} e^{i\phi_{jk}} |jk\rangle$

No! TP condition requires $\mathrm{tr}_1 |\psi\rangle\langle\psi| = \frac{1}{N}\mathbb{1}$

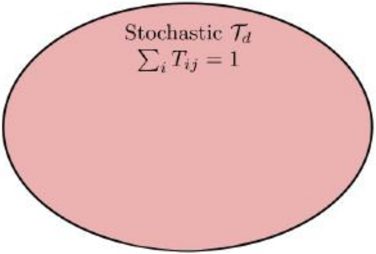
Example

$$T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\mathrm{tr}_1 |\psi\rangle\langle\psi| = |+\rangle\langle+|$$

Categories of classical transition matrix T



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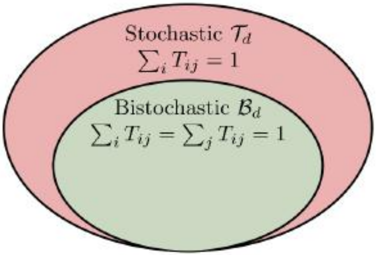

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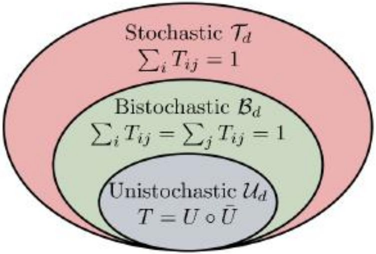

were $(A \circ B)_{jk} = A_{jk} B_{jk}$
denotes Hadamard product:

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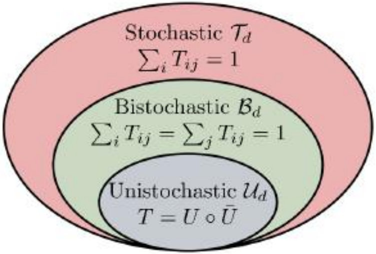

were $(A \circ B)_{jk} = A_{jk} B_{jk}$
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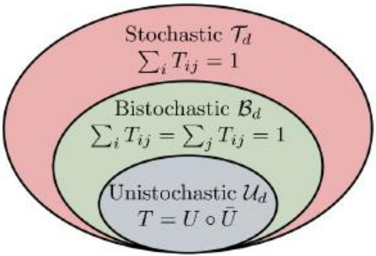
Schur example of bistochastic T
of order 3 which is not unistochastic

$$T = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \; X = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & e^{i\theta_{12}} & e^{i\theta_{13}} \\ e^{i\theta_{21}} & 0 & e^{i\theta_{23}} \\ e^{i\theta_{31}} & e^{i\theta_{32}} & 0 \end{bmatrix}$$

X is not unitary!

Categories of classical transition matrix T



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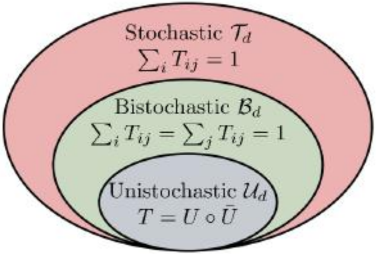

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X is not unitary!

were $(A \circ B)_{jk} = A_{jk} B_{jk}$ denotes Hadamard product:

Main result: **Proposition.**

Φ can be **coherified** to a unitary map $\Psi_U \iff T$ is **unistochastic**

Open **unistochasticity** problem: given **bistochastic** T, check if there is a unitary U such that $T_{ij} = |U_{ij}|^2$

Set of 2×2 **bistochastic** matrices, $B = \begin{bmatrix} 1-a & a \\ a & 1-a \end{bmatrix}$ with $a \in [0,1]$

$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P$

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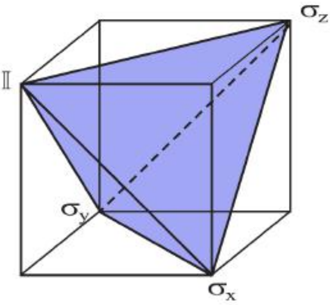
can be coherified into the **tetrahedron** of unital **Pauli channels**
as all bistochastic matrices of order $N = 2$ are **unistochastic** !



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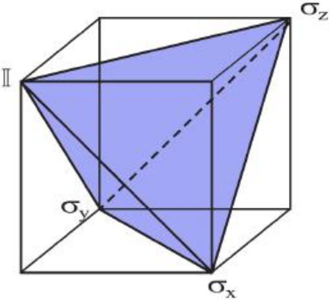
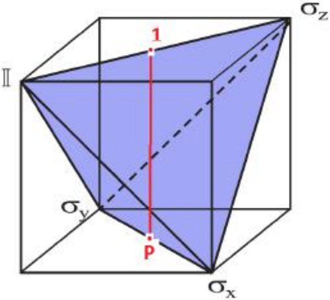
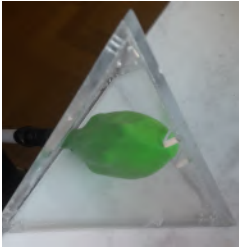




Three dimensional tetrahedron of **one–qubit**, unital, **Pauli channels**

decoheres to the 1-D interval $[0, 1]$ of *classical* **bistochastic matrices**

$\longrightarrow$



Optimal coherification of qubit channels

Classical action of a qubit channel:

$$T = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix} =: \begin{bmatrix} a & \tilde{b} \\ \tilde{a} & b \end{bmatrix}$$

Optimally coherified channel:

$$\Phi^{\mathcal{C}} = \Psi(U(\cdot)U^{\dagger})$$

with unitary

$$U = \frac{1}{\sqrt{a+\tilde{b}}} \begin{bmatrix} \sqrt{a} & -\sqrt{\tilde{b}} \\ \sqrt{\tilde{b}} & \sqrt{a} \end{bmatrix}$$

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and $\Psi(\cdot) = L_1(\cdot)L_1^{\dagger} + L_2(\cdot)L_2^{\dagger}$ with

$$L_1 = \begin{bmatrix} \sqrt{a+\tilde{b}} & 0 \\ 0 & 1 \end{bmatrix}, \; L_2 = \begin{bmatrix} 0 & 0 \\ \sqrt{b-a} & 0 \end{bmatrix}$$

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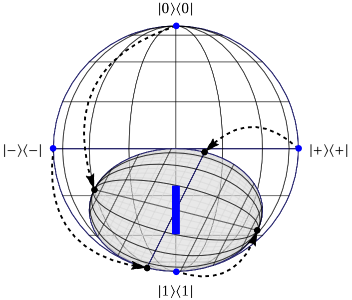

$$T = \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$$

Optimally coherified channel:

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with unitary

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Upper-bound for the degree of coherification

Optimising coherence of Φ with fixed $T \iff$ maximizing purity of J_Φ.

Majorization partial order: $p \succ q \iff \forall_k \sum_{j=1}^k p_j^\downarrow \geq \sum_{j=1}^k q_j^\downarrow$

Important because: $p \succ q \implies S(p) \leq S(q)$ and $p \cdot p \geq q \cdot q$

Look for $\mu^\succ(T)$ such that: $\forall \Phi$ with $diag(J_\Phi) = \frac{1}{d}|T\rangle\rangle$:

$\mu^\succ(T) \succ \lambda(J_\Phi)$

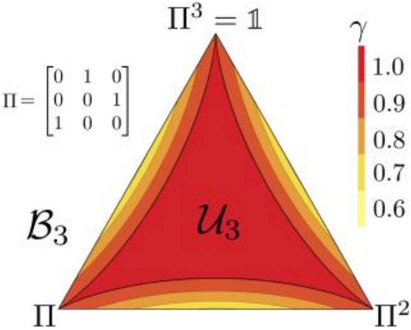
Why? To upper-bound C_e or C_2

Bistochastic classical transition matrix

For bistochastic T majorization upper-bound becomes trivial

$$[1, 0, \ldots, 0]^\top = \mu^{\succ} \succ \lambda(J_\Phi)$$

A non-trivial bound which describes the unistochastic-bistochastic boundary



Leads to bounds for the purity $\gamma = \mathrm{Tr}(J_\Phi)^2 \leq 1$ characterizing the coherified map Φ.

Perfectly distinguishable state coherifications

One can always optimally coherify state p

$$\rho = diag(p) \xrightarrow{\mathcal{C}} |\psi_j\rangle\langle\psi_j| \quad \text{with} \quad |\psi\rangle = \sum_k \sqrt{p_k} e^{i\phi_{jk}} |k\rangle$$

Classical states p related to $|\psi_j\rangle$ are the same and are **indistinguishable**.
However, if quantum states $|\psi_j\rangle$ are orthogonal they can be **distinguished**.

Question

How many perfectly distinguishable states with classical version p are there?

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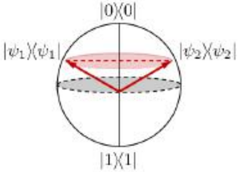

$\boldsymbol{p} = [3/4, 1/4]$

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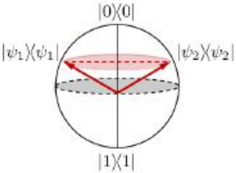

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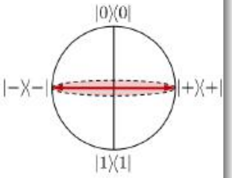


$\boldsymbol{p} = [1/2, 1/2]$

Necessary condition for M-distinguishability

Necessary condition

M perfectly distinguishable states of size N, with $\{\psi_i\}$ with $|\langle k|\psi_j\rangle|^2 = p_k,\ k = 1, \dots, N$ $\implies$ $\forall k : p_k \leq \frac{1}{M}$

Orthogonal $\{|\psi_j\rangle\}$ could form columns of unitary matrix

$$U = \begin{bmatrix} \sqrt{p_1}e^{i\phi_{11}} & \dots & \sqrt{p_1}e^{i\phi_{1N}} & \dots \\ \sqrt{p_2}e^{i\phi_{21}} & \dots & \sqrt{p_1}e^{i\phi_{2N}} & \dots \\ \vdots & \ddots & \vdots & \ddots \\ \sqrt{p_d}e^{i\phi_{d1}} & \dots & \sqrt{p_d}e^{i\phi_{dN}} & \dots \end{bmatrix}$$

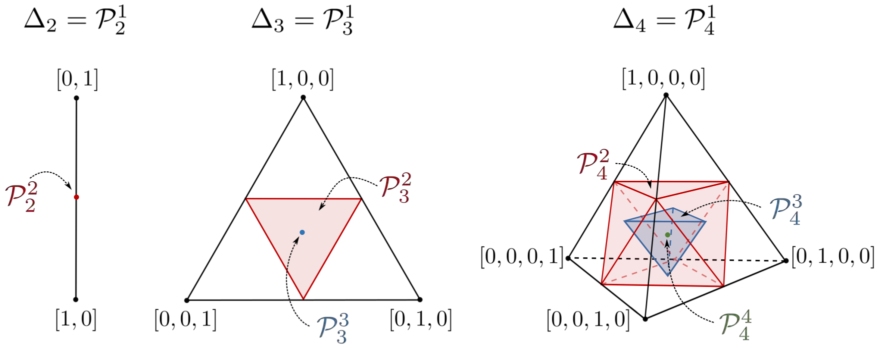
Corresponding unistochastic matrix:

$$U \circ \overline{U} = \begin{bmatrix} p_1 & \dots & p_1 & \dots \\ p_2 & \dots & p_2 & \dots \\ \vdots & \ddots & \vdots & \ddots \\ p_N & \dots & p_N & \dots \end{bmatrix}$$

But rows must sum to 1!

Quantum distinguishability of classical states

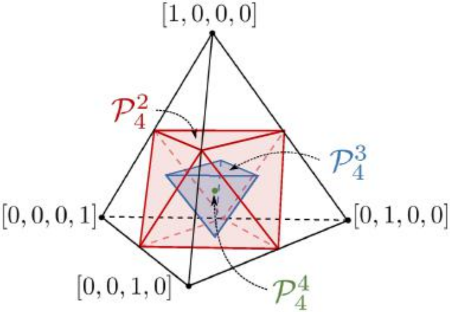
Set of classical states of size $N = 2, 3$ and 4 forms simplices Δ_{N-1}

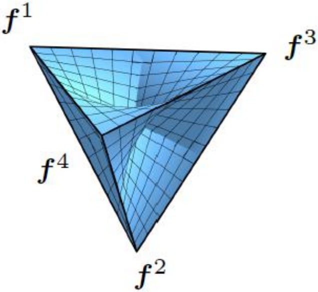


P_M^N denotes the subset of Δ_{N-1} containing M–distinguishable states

Example: $N = 4$ classical states (tetrahedron)

a) $m = 1$ - entire tetrahedron of classical states $\mathbf{p} = (p_1, p_2, p_3, p_4)$
b) The subset of classical states with $m = 2$ **distinguishable** quantum states defined by condition $p_{\max} \leq 1/2$ forms a dual red tetrahedron,
c) The subset with $m = 3$ **distinguishable** quantum states belongs to tetrahedron determined by $p_{\max} \leq 1/3$ and forms an object inside blue tetrahedron bounded with product states, $\alpha(a, 1-a) \times (b, 1-b)$,
d) Set with $m = 4$ forms the center, $p_* = \frac{1}{4}(1,1,1,1)$ - (Fourier matrix F_4)





Distinguishing channel coherifications

Channels $\{\Phi^{(j)}\}$ with fixed action T are perfectly distinguishable iff:

$\exists \rho_{AB}$ $\{\Phi^{(j)} \otimes \mathbb{1}(\rho_{AB})\}$ are perfectly distinguishable

If $\exists \rho$ $\{\Phi^{(j)}(\rho)\}$ are perfectly distinguishable then no entanglement is needed

Type of classical transition matrix T	**Number** of perfectly distinguishable channels	Requires **entanglement** ?
Unistochastic	d	No
Unistochastic	$d+1, \ldots, d^2$	Yes
Bistochastic	2	Yes
Such that $T_{jk} \leq \frac{1}{2}$	2	No

Concluding Remarks

- Hyper-Decoherence of a **quantum map** Φ to a **classical map** T determined by the diagonal of the Choi matrix J_Φ
  (a supermap $\Gamma(\Phi)$ yields the classical channel Φ_T)
- Measures of *coherence of a map* $\mathcal{C}(\Phi)$ proposed in analogy
  to the **coherence of a state** $\mathcal{C}(J_\Phi)$
- Idea of **coherification** of a state and a map (*Kannalsanierung*): the search for all preimages with respect to **decoherence**
- *Open questions*:
  * Are optimally coherified channels extremal?
  * Is the minimum output entropy equal to zero??
  * What is the number of perfectly distinguishable states (maps) which decohere to a given classical **state** / **map**

Based on **K. Korzekwa, S. Czachórski, Z. Puchała, K.Ż.**

a) New J. Phys. **20**, 043028 (2018)

and b) J. Phys. **A 52**, 475303 (2019)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Quanten
KANALSANIERUNG
MPQ
Garching
MPQ

A short message to a **theoretical physicist** :

A short message to a **theoretical physicist** :

From time to time it is good to look through the window,
to observe the **real world** **outside**,

so it is also good to **wash** it from time to time ...

Bench commemorating discussion between

Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)

Sculpture: Stefan Dousa Fot. Andrzej Kobos

Opened in Planty Garden, Cracow, Oct. 14, 2016